CANONICAL STABILITY MATRICES FOR LINEAR TIME-INVARIANT SYSTEMS AND THEIR APPLICATIONS

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MASTER OF TECHNOLOGY IN ELECTRICAL ENGINEERING

by

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Department of Electrical Engineering

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CERTIFICATE

Certified that this work on "Canonical Stability Matrices for Linear Time-Invariant Systems and Their Applications" has been carried out under our supervision and that this has not been submitted elsewhere for a degree.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SYNOPSIS

CANONICAL STABILITY MATRICES FOR LINEAR TIME-INVARIANT SYSTEMS AND THEIR APPLICATIONS

Ramamoorthy Viswanathan

Schwarz in 1956 introduced a method of checking the stability of a linear time-invariant system described by the vector differential equation

$$\underline{\dot{x}} = \underline{A}\ \underline{x}$$

This method requires transforming the matrix $\underline{A}$ through a series of elementary transformations to a canonical matrix $\underline{B}$ given by

$$\underline{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_2 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -b_{n-1} & -b_n \end{bmatrix}$$

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Using the concept of continued fractions, Schwarz

showed that the given system is asymptotically stable if and only if all the b_i's, $i=1,2,\dots,n$ of $\underline{B}$ are positive and that the number of stable eigenvalues is equal to the number of positive terms in the sequence b_n, $b_n b_{n-1}, \dots, (b_n b_{n-1} b_{n-2} \dots b_1)$. Recently in the control systems literature there has been a great deal of interest shown towards finding uses for the above Schwarz canonical matrix. Through this canonical matrix, Liapunov function for the system can be quite simply constructed and a certain class of performance measures can be evaluated. It has been used to establish a link between the second method of Liapunov and the Routh-Hurwitz stability criterion.

In this thesis, three different- but similar forms of the Schwarz matrix are identified and the similarity transformations between them are determined. The problem of transforming a given system described by

$$\dot{\underline{x}} = \underline{A}\ \underline{x}$$

to one in Schwarz form

$$\dot{\underline{y}} = \underline{B}\ \underline{y}$$

is dealt with systematically. When $\underline{A}$ is in phase-variable canonical form, the associated transformation is shown to be quite simple and it requires the

knowledge of the elements of the Routh-array of the system. The transformation between the Schwarz form and its Jordan form is also determined and it turns out to be quite analogous to the one between phase-variable form and its Jordan form. Schwarz canonical form for controllable multivariable systems is proposed. This consists of a set of coupled subsystems each of which being in Schwarz form. Transforming any arbitrary controllable multivariable system to this canonical form is also discussed.

A network interpretation is given to the three forms of the Schwarz matrix. By a proper choice of state variables, the state equations of a resistively terminated L-C ladder network are shown to be in any one of these forms. Also, given a system in Schwarz form, a procedure is given to evaluate the components of the associated resistively terminated L-C ladder network. Thus such a network interpretation is shown to be useful both for analysis and synthesis problems. Network interpretation for multivariable systems in Schwarz canonical form is also dealt with.

An important use of the Schwarz canonical form in simplifying large dynamic systems is also investigated in this thesis. Problems involving large dynamic systems demand complex computational schemes and thus

it is desirable to make reasonable approximations to simplify such large systems. The simplification method proposed in this thesis first transforms the given 'large' system to one in Schwarz form, compares the elements b_i's of the Schwarz form and thus determines whether any simplification is possible or not and finally carries out the actual simplification. Several examples are considered for illustrating the method. Some difficulties encountered by this method are also pointed out. The major advantage of this method is that it does not require the computation of the system eigenvalues and eigenvectors.

The Schwarz matrix is used to derive a new canonical matrix for linear time-invariant discrete-time systems. This new canonical matrix is the bilinear transform of the Schwarz matrix. Liapunov functions for linear time-invariant discrete systems can be quite simply constructed through the use of the proposed canonical matrix whose elements, as in the case of the Schwarz matrix for the continuous-time systems, contain the stability information of the system. An attempt is made to establish a link between this canonical matrix and the Jury table test. Several methods of transforming a given discrete-time system to one in this canonical form are also discussed.

Several unsolved problems are suggested for further research.

CHAPTER - I

INTRODUCTION

Canonical forms for system description are used extensively in both analysis and synthesis of control systems. The significance of these forms is fairly obvious. Typically in a synthesis problem, the number of parameters to be designed becomes less (minimal) with the use of canonical forms. Analysis of systems with such description becomes simpler, as elegant analytical results can be obtained in such cases. To quote examples, in control system literature, the phase-variable and the Jordan canonical forms are well known.

The Jordan form displays the system eigenvalues explicitly and hence is ideally suited for the analysis problem. However, given a system in any arbitrary form, transforming this to the Jordan form involves the computation of the system eigenvalues and eigenvectors and thus becomes computationally difficult especially when the system size is large. Further, the problem of degeneracy with repeated eigenvalues makes such a transformation still more difficult. The phase-variable form, on the other hand, is quite simple and the corresponding transformation is comparatively easy. But, the elements of the phase-variable form do not have as much information (for instance, stability of the system) as those of the Jordan form. This investigation concerns with a canonical form which has both simplicity in terms

of the associated transformation and "enough" information about the system behaviour "stored" in its elements.

Schwarz[1] in 1956 introduced, for stability investigation of linear time-invariant continuous-time systems, a new canonical form which was later popularized by Kalman and Bertram[2] and which is now commonly known as the Schwarz canonical form. This canonical form has been used to construct Liapunov functions [2] , to prove the Routh-Hurwitz stability criterion through the second method of Liapunov [3] , to derive optimum transfer functions [4] , to evaluate system performance measures [4,5,6] and to solve the Liapunov matrix equation [8,9,10,11] .

The Schwarz canonical form for linear systems is of central interest in this investigation. Three different- but similar ~~simpler~~- forms of the Schwarz canonical matrix are identified and the similarity transformations between these forms are established. Given a system in phase-variable form, transforming this to the Schwarz form has been achieved quite recently [13,14] . This transformation process has been further simplified in this investigation. When the given system is in any arbitrary form, the associated transformation to the Schwarz form is described. A Vandermonde-like transformation matrix which takes the system description from Schwarz to Jordan form has been found out. Schwarz canonical form for multivariable systems has been proposed. This result has been obtained by

extending the recent work of Luenberger [15]. A simple and interesting network interpretation is given to the Schwarz canonical form both for single-variable and multivariable systems. It is hoped that this network interpretation would facilitate both analysis and synthesis problems. Added to these theoretical investigations, an important practical use for the Schwarz form in simplifying large dynamic systems has been suggested and the simplification procedure has been investigated in detail. Several examples have been considered for illustrating the method. A new canonical form for linear time-invariant discrete-time systems has been proposed. This canonical form has been obtained through the bilinear transformation of the Schwarz form. It has been shown to be useful in constructing Liapunov functions. The properties of this canonical form have been thoroughly investigated. A number of unsolved problems are suggested for further research.

CHAPTER - II

SCHWARZ CANONICAL FORM

2.1 INTRODUCTION

In this chapter, Schwarz canonical form for linear time-invariant continuous-time systems is introduced. Some of the results regarding this canonical form which are already available in the literature are stated.

In section 2, the Schwarz canonical matrix is developed for systems having characteristic polynomials with real coefficients. Three different (but similar) forms of the Schwarz matrix and the transformations between them are also given in this section. Section 3 discusses various properties and uses of the Schwarz canonical matrix.

2.2 DEVELOPMENT OF THE SCHWARZ CANONICAL MATRIX [1]

Consider a system described by

$$\dot{\underline{x}} = \underline{A}\,\underline{x} \tag{2.2.1}$$

Let the characteristic equation of (2.2.1) be given by:

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \tag{2.2.2}$$

The polynomial

$$Q(\lambda) = a_1 \lambda^{n-1} + a_3 \lambda^{n-3} + \ldots \tag{2.2.3}$$

is known as the alterant of $P(\lambda)$ in (2.2.1) [16].

The test fraction of $P(\lambda)$ is defined as:

$$\frac{Q(\lambda)}{P(\lambda)} = \cfrac{1}{r_1\lambda + 1 + \cfrac{1}{r_2\lambda + \cfrac{1}{r_3\lambda + \cfrac{1}{r_4\lambda + \cdots}}}} \qquad (2.2.4)$$

where

$$r_m = \frac{p_{(m-1)\ (m-1)}}{p_{m\ m}}, \quad (m=1,2,\ldots,n) \qquad (2.2.5)$$

$$\left.\begin{aligned} p_{oj} &= \begin{cases} 0 & j \text{ odd} \\ a_j & j \text{ even} \quad \text{with } a_o = 1 \end{cases} \\ p_{1j} &= \begin{cases} 0 & j \text{ even} \\ a_j & j \text{ odd} \end{cases} \\ & (j = 0,1,2,\ldots,n) \end{aligned}\right\} \qquad (2.2.6)$$

$$p_{ij} = p_{(i-2)j} - \gamma_{i-1} \quad p_{(i-1)(j+1)}, \; (i,j = 2,3, \quad)$$

Substituting in (2.2.4)

$$\frac{1}{r_1} = b_n, \quad \frac{1}{r_k\, r_{k-1}} = b_{n-k+1}, \quad (k = 2,3,\ldots,n) \qquad (2.2.7)$$

results in

$$\frac{Q(\lambda)}{P(\lambda)} = \cfrac{-b_n}{-b_n - \lambda + \cfrac{b_{n-1}}{-\lambda + \cfrac{b_{n-2}}{-\lambda + \cfrac{b_{n-3}}{\ddots + \cfrac{b_1}{\lambda}}}}} \qquad (2.2.8)$$

By applying a theorem in continued fraction 160 , equation (2.2.8) yields

$$P(\lambda) = \begin{vmatrix} -b_n-\lambda & b_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ -1 & -\lambda & b_{n-2} & 0 & \cdots & 0 & 0 \\ 0 & -1 & -\lambda & b_{n-3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & b_1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & -\lambda \end{vmatrix} \qquad (2.2.9)$$

i.e.,

$$P(\lambda) = \left| \underline{S}_3 - \lambda \underline{I} \right| \qquad (2.2.10)$$

where

$$\underline{S}_3 = \begin{bmatrix} -b_n & b_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & b_{n-2} & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & b_{n-3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \qquad (2.2.11)$$

$\underline{S}_3$ in (2.2.11) is similar to $\underline{A}$ in (2.2.1) and it is one of the three forms of the Schwarz matrix. The other two forms are denoted by $\underline{S}_1$ and $\underline{S}_2$ and they are given as:

$$\underline{S}_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -b_1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -b_2 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -b_{n-1} & -b_n \end{bmatrix} \tag{2.2.12}$$

$$\underline{S}_2 = \begin{bmatrix} 0 & b_1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & b_2 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & -b_n \end{bmatrix} \tag{2.2.13}$$

Observe that $\underline{S}_2$ and $\underline{S}_3$ are given in [1] whereas Kalman and Bertram [2] have made use of the form $\underline{S}_1$. In this report also, $\underline{S}_1$ is mostly used and the letter $\underline{B}$ is interchangeably used to denote $\underline{S}_1$. Whenever we refer to the Schwarz matrix without any qualification, we mean $\underline{S}_1$ only.

The similarity transformations between $\underline{S}_1$ and $\underline{S}_2$ and between $\underline{S}_1$ and $\underline{S}_3$ will now be considered. Let

$$\underline{S}_2 = \underline{T}_2^{-1} \, \underline{S}_1 \, \underline{T}_2 \tag{2.2.14}$$

and

$$\underset{\sim}{S}_3 = \underset{\sim}{T}_3^{-1} \; \underset{\sim}{S}_1 \; \underset{\sim}{T}_3 \qquad (2.2.15)$$

Then

$$\left.\begin{aligned}
(\underset{\sim}{T}_2)_{12} &= 1 \\
(\underset{\sim}{T}_2)_{21} &= -1 \\
(\underset{\sim}{T}_2)_{ii} &= 0, \; i=1,2,\ldots,n-1 \\
(\underset{\sim}{T}_2)_{nn} &= -\prod_{k=2}^{n} (-b_k) \\
&\left.\begin{aligned}
(\underset{\sim}{T}_2)_{i(i+1)} &= -\prod_{k=2}^{i} (-b_k) \\
(\underset{\sim}{T}_2)_{(i+1)i} &= -(\underset{\sim}{T}_2)_{i(i+1)}
\end{aligned}\right\} i=2,3,\ldots,n-1
\end{aligned}\right\} \qquad (2.2.16)$$

$\underset{\sim}{T}_2$ as given in (2.2.16) is tridiagonal. Except for the element $(\underset{\sim}{T}_2)_{nn}$, $\underset{\sim}{T}_2$ is skew-symmetric. For n=4,

$$\underset{\sim}{T}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -(-b_2) & 0 \\ 0 & (-b_2) & 0 & -(-b_2)(-b_3) \\ 0 & 0 & (-b_2)(-b_3) & -(-b_2)(-b_3)(-b_4) \end{bmatrix}$$

The transformation matrix $\underset{\sim}{T}_3$ is a cross-diagonal matrix (i.e., the only nonzero elements lie on the cross-diagonal) and is given by:

$$\left.\begin{aligned}
(\underset{\sim}{T}_3)_{ij} &= 0, \; j \neq n-i+1 \\
(\underset{\sim}{T}_3)_{i(n-i+1)} &= (-1)^{n-i+1}, \; (i=1,2,\ldots,n)
\end{aligned}\right\} \qquad (2.2.17)$$

For $n = 4$,

$$\underline{T}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

In chapter IV, the concept of network interpretation to the Schwarz form is effectively used to derive a simpler transformation $\underline{M}_2$ between $\underline{S}_1$ and $\underline{S}_2$.

2.3 PROPERTIES AND USES OF THE SCHWARZ CANONICAL FORM

Consider a system described by

$$\dot{\underline{y}} = \underline{B}\,\underline{y} + \underline{f}\,u \tag{2.3.1}$$

where $\underline{B}$ is as in (2.2.12) and

$$\underline{f} = (0, 0, \ldots, 0, 1)^T \tag{2.3.2}$$

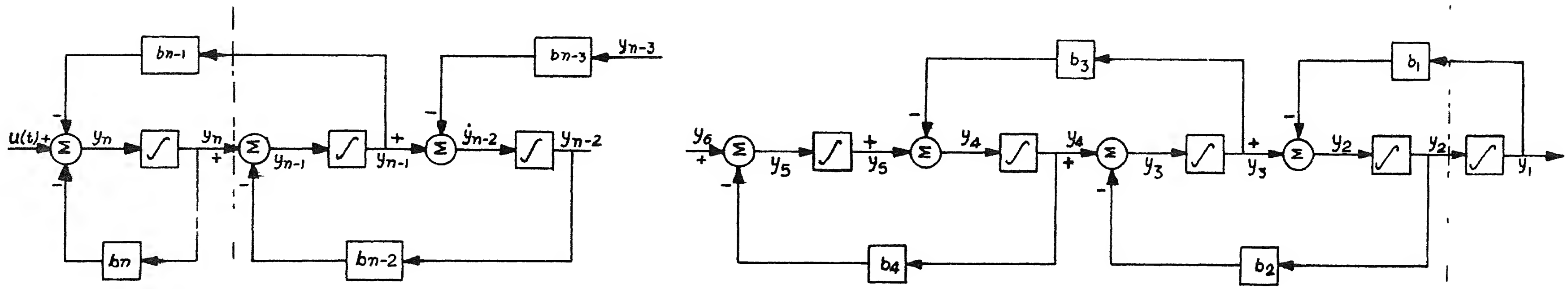
Elaborately writing the differential equations in (2.3.1)

$$\left.\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -b_1 y_1 + y_3 \\
\dot{y}_3 &= -b_2 y_2 + y_4 \\
\dot{y}_4 &= -b_3 y_3 + y_5 \\
&\vdots \\
\dot{y}_{n-3} &= -b_{n-4}\, y_{n-4} + y_{n-2} \\
\dot{y}_{n-2} &= -b_{n-3}\, y_{n-3} + y_{n-1} \\
\dot{y}_{n-1} &= -b_{n-2}\, y_{n-2} + y_n
\end{aligned}\right\} \tag{2.3.3}$$

These differential equations can be equivalently represented by means of a simulation diagram as in Figure (2.3.1) (In this Figure, n is taken as an even number. A similar diagram may be drawn for n odd case also). Except for the first and the last differential equations in (2.3.3), others have the same pattern. The simulation diagram also exhibits such a pattern between the dotted lines [See Figure (2.3.1)] i.e., between the first and last integrators. Such simulation diagram may be useful for computing the response of a system in Schwarz form by using analog computer or digital computer (For instance, "Pactolus" program may be used). In fact, this has been used for the example systems in chapter V.

By modifying the results in continued fraction method [16], Schwarz [1] has obtained the following: The system in (2.2.1) is asymptotically stable if and only if all the b_i's $(i=1,2,\ldots,n)$ in (2.2.11) or (2.2.12) or (2.2.13) are positive. The number of eigenvalues of $\underline{A}$ in (2.2.1) with negative real parts is given by the number of positive terms in the sequence b_n, $b_n b_{n-1}$, $b_n b_{n-1} b_{n-2}, \ldots,$ $b_n b_{n-1} b_{n-2} \ldots b_1$. This is an important result and for future reference, let us call this as Schwarz theorem. Kalman and Bertram [2] have proved this theorem through the second method of Liapunov. Consider a system

$$\dot{\underline{y}} = \underline{B}\,\underline{y} \qquad (2.3.4)$$



SIMULATION DIAGRAM FOR A SYSTEM IN SCHWARZ CANONICAL FORM

FIGURE (2 3 1)

and its Liapunov function

$$V = \bar{y}^T \underline{L} \underline{y} \qquad (2.3.5)$$

where

$$\underline{L} = \text{diag}\left\{(b_n b_{n-1},\ldots,b_1),\ (b_n b_{n-1}\ldots b_2),\ldots,\ b_n b_{n-1}, b_n\right\}. \qquad (2.3.6)$$

Then

$$\dot{V} = -2b_n^2\, y_n^2 \qquad (2.3.7)$$

which is negative semidefinite . Also $\dot{V}$ cannot be identically zero unless $y_i = 0$ $(i=1,2,\ldots,n)$ [4] . Thus from the second method of Liapunov, the system in (2.3.4) is asymptotically stable if and only if $\underline{L}$ in (2.3.6) is positive definite. Hence Schwarz theorem is proved.

Parks [3,4] has shown that the Schwarz elements b_i's $(i=1,2,\ldots,n)$ are related to the Hurwitz determinants Δ_i's $(i=1,2,\ldots,n)$ as under:

$$b_i = \frac{\Delta_{n-i+1}\ \Delta_{n-i-2}}{\Delta_{n-i}\ \Delta_{n-i-1}}, \quad (i=1,2,\ldots,n) \qquad (2.3.8)$$

with $\Delta_k = 1$, $k \leq 1$. It may be easily verified that r_m in (2.2.5) is expressed in terms of the elements in the first column of the Routh array, R_{1j} $(j=1,2,\ldots,n+1)$, of (2.2.2) as follows:

$$r_m = \frac{R_{m1}}{R_{(m+1)1}}, \quad m = 1,2,\ldots,n \qquad (2.3.9)$$

with $R_{11} = 1$. From (2.2.7) and (2.3.9) it is not difficult to see that

$$b_i = \frac{R_{(n-i+2)1}}{R_{(n-i)1}}, \quad i = 1,2,\ldots,n \tag{2.3.10}$$

with $R_{k1} = 1$ when $k \leq 1$. From (2.3.8), (2.3.10) and Schwarz theorem or the result of the second method of Liapunov (i.e., $\underline{L} > \underline{0}$), it is inferred that all the first column elements of the Routh array or all the Hurwitz determinants must be positive. This is how Parks [3,4] has established the link between the second method of Liapunov and the Routh-Hurwitz stability criterion. He has also used the Schwarz canonical form to derive 'optimum transfer functions'. For details, refer to [4].

From equations (2.3.5), (2.3.6) and (2.3.7) it is clear that the Schwarz canonical form can be used for constructing Liapunov function for a system. Suppose the given system has description as in (2.2.1), Apply a transformation*

$$\underline{y} = \underline{P}\ \underline{x} \tag{2.3.11}$$

---

* $\underline{A}$ in (2.2.1) may be in any arbitrary form or in a special case in phase-variable form, According to the notation used in latter chapters, $\underline{P}$ is the transformation matrix such that $\underline{B} = \underline{P}\,\underline{A}\,\underline{P}^{-1}$ where $\underline{A}$ is in phase-variable form. Instead if $\underline{A}$ is in any arbitrary form, $\underline{H}$ is the transformation matrix such that $\underline{B} = \underline{H}^{-1}\,\underline{A}\,\underline{H}$. However, in (2.3.11) we just use $\underline{P}$ without assuming the form of $\underline{A}$.

Then the Liapunov function for the system in (2.2.1) can be easily written as

$$V = \underline{x}^T \underline{P}^T \underline{L} \underline{P} \underline{x} \tag{2.3.12}$$

in view of (2.3.5) and (2.3.11). The transformation (2.3.11) will be discussed elaborately in chapter III.

The Schwarz canonical form can be used to evaluate the system performance measures of the type to be given below [5,6]. But this requires that the system description is to be in Schwarz form [See equation (2.3.4)] . Assume that the system is asymptotically stable. Consider now the performance measure

$$I = \int_0^\infty y_n^2 \, dt \tag{2.3.13}$$

Integrating (2.3.7) between o and ∞ with respect to time results in

$$I = -\frac{1}{2b_n^2} \left(V(\infty) - V(o) \right) \tag{2.3.14}$$

Noting that for an asymptotically stable system, $V(\infty) = 0$ and substituting (2.3.5) in (2.3.14)

$$I = \int_0^\infty y_n^2 \, dt = \frac{1}{2b_n^2} \underline{y}^T(o) \, \underline{L} \, \underline{y}(o) \tag{2.3.15}$$

In reference [5] , this method has been extended to evaluate performance measures of the type

$$J_k = \int_0^\infty t^k y_n^2 \, dt \, , \quad k = 1,2,\ldots \tag{2.3.16}$$

Recently [7,8,9,10,11,12] there has been a great deal of interest shown towards solving the Liapunov matrix equation

$$\underline{A}^T \underline{K} + \underline{K}\,\underline{A} = -\underline{G} \tag{2.3.17}$$

where $\underline{K} > \underline{0}$ and $\underline{G} \geq \underline{0}$. The solution of the equation (2.3.17) may be useful for calculating functionals of motion [17], estimating overshoots and settling times [18] and deriving pseudo-optimal control policies for the vector $\underline{u}$ which will return the system

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{G}\,\underline{u} \tag{2.3.18}$$

to equilibrium as quickly as possible following an initial disturbance [18]. Schwarz canonical form has been used to determine the solution of (2.3.17) quite simply. For details, refer to [8,9,10,11].

2.4 CONCLUSION

Some of the uses for the Schwarz canonical form discussed in this chapter rely upon our ability to compute the transformation matrix $\underline{P}$ [See equation (2.3.11)]. This transformation is discussed elaborately in the next chapter. In chapter V, it is shown that the Schwarz canonical form can be used to simplify large dynamic systems. A possible topic for further investigation is the modeling of the physical systems to get the system description directly in Schwarz form.

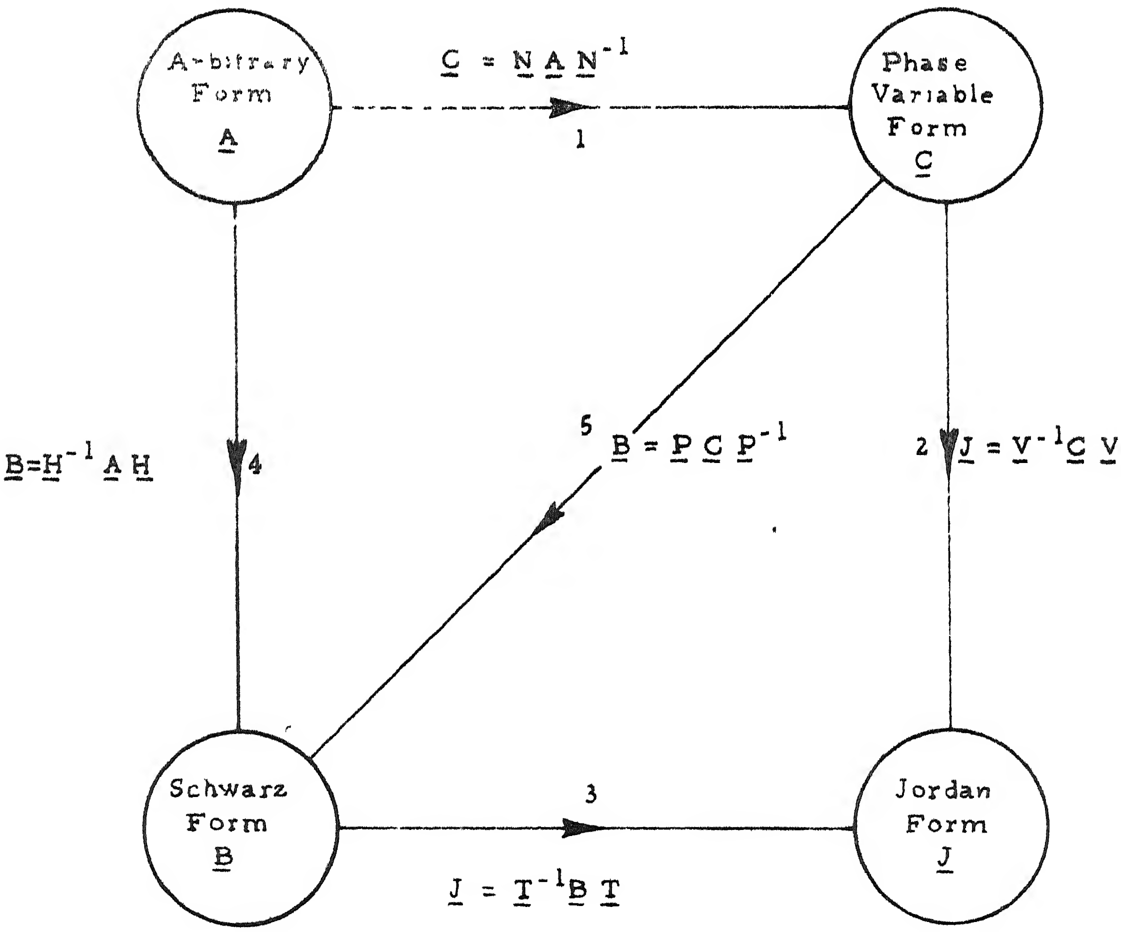
CHAPTER - III

TRANSFORMATIONS ASSOCIATED WITH SCHWARZ CANONICAL FORM

3.1 INTRODUCTION

In the last chapter, some of the uses for the Schwarz canonical form were discussed. However, it was not stated as to how a system in any arbitrary form could be transformed to the Schwarz canonical form. This problem is treated in this chapter. Schwarz [1] used a series of elementary transformations to transform any given matrix to its Schwarz form. However, no simple similarity transformation like the one between the arbitrary form and the phase-variable form [19] was reported till recently. Recently there has been a great deal of interest shown towards finding such a transformation to the Schwarz canonical form [13,14,20,21] .

The contents of this chapter will now be briefly stated with the help of the Figure (3.1.1). Section 2 discusses the transformation between the phase-variable form $\underline{C}$ and the Schwarz form $\underline{B}$ i.e., the link No.5 in Figure (3.1.1) . The elements of the corresponding transformation matrices $\underline{P}$ and $\underline{P}^{-1}$ can be explicitly expressed in terms of either the elements R_{ij}'s of the Routh array or the elements b_i's of $\underline{B}$. A critical discussion on these methods of determining $\underline{P}$ and $\underline{P}^{-1}$ is also contained in section 2. Section 3 deals with



TRANSFORMATIONS TO CANONICAL FORMS

FIGURE (3.1.1)

the transformation from the arbitrary form $\underline{A}$ to the Schwarz form $\underline{B}$ (link No.4). The transformation between the Schwarz and the Jordan forms is the subject of section 4 (link No.3). An elegant method of determining $\underline{N}$ (link No.1) is given in [20]. Of course, the transformation in link No.2 is achieved through the familiar Vandermonde matrix, $\underline{V}$. In section 5, Schwarz canonical form for multivariable systems is proposed and the transformation to this canonical form is also discussed.

3.2 THE TRANSFORMATION FROM THE PHASE-VARIABLE FORM TO THE SCHWARZ FORM

Consider a system described by

$$\dot{\underline{x}} = \underline{C}\ \underline{x} \tag{3.2.1}$$

where $\underline{C}$ is in phase-variable form. It is desired to find a transformation

$$\underline{y} = \underline{P}\ \underline{x} \tag{3.2.2}$$

which takes the system description from (3.2.1) to

$$\dot{\underline{y}} = \underline{B}\ \underline{y} \tag{3.2.3}$$

where $\underline{B}$ is in Schwarz form. In other words,

$$\underline{B} = \underline{P}\ \underline{C}\ \underline{P}^{-1} \tag{3.2.4}$$

It has been found [13,14] that the elements of $\underline{P}$ and $\underline{P}^{-1}$ can be expressed in terms of the elements of the Routh array, R_{ij}'s, of the system. The results will now be stated. The matrices $\underline{P}$ and $\underline{P}^{-1}$ have the following interesting properties:

1. They are lower triangular matrices. That is, all the elements above the main diagonal of $\underline{P}$ and $\underline{P}^{-1}$ are zeros.
2. All the elements on the main diagonal of $\underline{P}$ and $\underline{P}^{-1}$ are equal to 1.
3. All the odd-ordered subdiagonals of $\underline{P}$ and $\underline{P}^{-1}$ contain zeros.
4. The even-ordered subdiagonals of $\underline{P}$ and $\underline{P}^{-1}$ contain nonzero elements and these elements are determined from the elements R_{ij}'s of the Routh array of the system.
5. If $\underline{P}_n$ is the $m \times n$ transformation matrix $\underline{P}$, then $\underline{P}_k$, $k = 1,2,\ldots,n-1$ (That is, $\underline{P}_k$ is the transformation matrix for a system of order k) are the submatrices located at the lower right corner of $\underline{P}_n$. Such submatrices will be hereafter referred to as the ascending principal minors of $\underline{P}_n$ of appropriate order. The matrix $\underline{P}_n^{-1}$ also has this property.

Determination of the elements on the even-ordered subdiagonals of $\underline{P}$ and $\underline{P}^{-1}$ will be discussed next. To begin with, the results obtained in references [13] and [14] will be stated briefly. Then it will be shown how the same results can be put in a simpler and more useful form.

Consider first the matrix $\underline{P}$. For convenience,

let $n = 8$. The $\underline{P}$ matrix for this case is shown below:

$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{R_{72}}{R_{71}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{R_{62}}{R_{61}} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{R_{53}}{R_{51}} & 0 & \frac{R_{52}}{R_{51}} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{R_{43}}{R_{41}} & 0 & \frac{R_{42}}{R_{41}} & 0 & 1 & 0 & 0 \\ \frac{R_{34}}{R_{31}} & 0 & \frac{R_{33}}{R_{31}} & 0 & \frac{R_{32}}{R_{31}} & 0 & 1 & 0 \\ 0 & \frac{R_{24}}{R_{21}} & 0 & \frac{R_{23}}{R_{21}} & 0 & \frac{R_{22}}{R_{21}} & 0 & 1 \end{bmatrix} \qquad (3.2.5)$$

Observe from (3.2.5) that the elements of $\underline{P}$ on the second subdiagonal are computed by dividing the corresponding elements in the second column by the elements in the first column of the Routh array. In fact, the last element in this second subdiagonal is $\frac{R_{22}}{R_{21}}$, the element just before this is $\frac{R_{32}}{R_{31}}$... etc. and the first element in this subdiagonal

is $\frac{R_{(n-1)2}}{R_{(n-1)1}}$ (in this case it is $\frac{R_{72}}{R_{71}}$). Similarly, for computing the fourth subdiagonal elements of $\underline{P}$, Routh array elements in the third and first columns are to be considered and for the k-th subdiagonal (k even) elements of $\underline{P}$, Routh array columns (k/2 + 1) and 1 are to be considered.

Consider next $\underline{P}^{-1}$. The elements on the second subdiagonal of $\underline{P}^{-1}$ are (from the lower right corner to the upper left corner) given as.

$$-\frac{R_{22}}{R_{21}},\ -\frac{R_{32}}{R_{31}},\ -\frac{R_{42}}{R_{41}},\ \ldots,\ -\frac{R_{(n-1)2}}{R_{(n-1)1}}$$

Note that these are same as the elements on the second subdiagonal of $\underline{P}$ with a negative sign. From the lower right corner to the upper left corner, the elements on the fourth subdiagonal of $\underline{P}^{-1}$ are the 2 x 2 determinants

$$\begin{vmatrix} \frac{R_{42}}{R_{41}} & 1 \\ \frac{R_{23}}{R_{21}} & \frac{R_{22}}{R_{21}} \end{vmatrix},\ \begin{vmatrix} \frac{R_{52}}{R_{51}} & 1 \\ \frac{R_{33}}{R_{31}} & \frac{R_{32}}{R_{31}} \end{vmatrix},\ \begin{vmatrix} \frac{R_{62}}{R_{61}} & 1 \\ \frac{R_{43}}{R_{41}} & \frac{R_{42}}{R_{41}} \end{vmatrix},\ \ldots,\ \begin{vmatrix} \frac{R_{(n-1)2}}{R_{(n-1)1}} & 1 \\ \frac{R_{(n-3)3}}{R_{(n-3)1}} & \frac{R_{(n-3)2}}{R_{(n-3)1}} \end{vmatrix}$$

which are illustrated in the following:

$$\underline{P} = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdots & R_{62}/R_{61} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & R_{52}/R_{51} & 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdots & R_{43}/R_{41} & 0 & R_{42}/R_{41} & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & R_{33}/R_{31} & 0 & R_{32}/R_{31} & 0 & 1 & 0 \\ \cdot & \cdots & R_{24}/R_{21} & 0 & R_{23}/R_{21} & 0 & R_{22}/R_{21} & 0 & 1 \end{bmatrix}$$

The elements on the sixth subdiagonal of $\underline{P}^{-1}$ are 3×3 determinants which can be obtained from $\underline{P}$ in a similar manner [14] . Note that this procedure of determining $\underline{P}^{-1}$ involves computation of determinants of order upto $\left[\frac{n-1}{2}\right]$ where the square brackets refer to the integer operation, i.e., $[x]$ means the largest integer contained in x.

These results will now be put in a more compact and useful form. Consider $\underline{P}$ first.

$$\left.\begin{aligned} \underline{P} &= [p_{ij}]_{n \times n} \\ p_{ij} &= \begin{cases} 1 & , \quad i = j \\ 0 & , \quad j > i \end{cases} \\ p_{(s+i)s} &= \begin{cases} 0 & i \text{ odd} \\ \dfrac{R_{(n+2-i-s)(i/2+1)}}{R_{(n+2-i-s)1}} & i \text{ even} \end{cases} \\ & \qquad i = 1,2,\ldots,n-1 \\ & \qquad s = 1,2,\ldots,n-1 \end{aligned}\right\} \qquad (3.2.6)$$

Note that in (3.2.6) $\underline{P}$ is represented in a quite simple and compact form and that $p_{(s+i)s}$, $s = 1,2,\ldots,n-i$ in (3.2.6) are the elements along the i-th subdiagonal of $\underline{P}$.

Consider next $\underline{P}^{-1}$. For convenience, let $\underline{Q} = \underline{P}^{-1}$. Then

$$\left.\begin{aligned}
\underline{Q} &= \left[q_{ij}\right]_{n \times n} \\
q_{ij} &= \begin{cases} 1 & i = j \\ 0 & j > i \end{cases} \\
q_{(s+i)s} &= \begin{cases} 0 & i \text{ odd} \\ -\displaystyle\sum_{m=1}^{i/2} \frac{R_{(n-s-i+2)(i/2+2-m)}}{R_{(n-s-i+2)1}} \, q_{(2m+s-2)s} & i \text{ even} \\ i = 1,2,\ldots,n-1 & \\ s = 1,2,\ldots,n-i & \end{cases}
\end{aligned}\right\} \quad (3.2.7)$$

The following comments are now in order. First, (3.2.7) gives a compact representation of $\underline{P}^{-1}$. Second, as before $q_{(s+i)s}$, $s = 1,2,\ldots,n-i$ are the elements on the i-th subdiagonal of $\underline{P}^{-1}$. Third, computation of determinants is avoided in this representation. The computational difficulty is now reduced to a large extent because in the process of computation, the knowledge of the previously computed elements is used effectively [See equation(3.2.7)]. Equation (3.2.7) is especially useful for machine computation.

In the rest of this section, it is aimed to determine $\underline{P}$ and $\underline{P}^{-1}$ in a still simpler way. First, the elements of

$\underline{P}$ and $\underline{P}^{-1}$ will be expressed in terms of the Schwarz elements b_i's $(i=1,2,\ldots,n)$.

Consider the elements $p_{i(i-2)}$, $i = 3,4,\ldots,n$ of the second lower diagonal of $\underline{P}$. They are given by

$$p_{i(i-2)} = \frac{R_{(n-i+2)2}}{R_{(n-i+2)1}}, \quad (i=3,4,\ldots,n) \tag{3.2.8}$$

It is quite simple to see that

$$\begin{aligned} R_{(n-1)2} &= R_{(n+1)1} \\ R_{(n-i+2)2} &= \frac{R_{(n-i+3)2}\ R_{(n-i+2)1}}{R_{(n-i+3)1}} + R_{(n-i+4)1} \\ & \qquad (i=4,5,\ldots,n) \end{aligned} \tag{3.2.9}$$

Referring to chapter II,

$$b_i = \frac{R_{(n-i+2)1}}{R_{(n-i)1}} \tag{3.2.10}$$

with $R_{kl} = 1$ when $k \leq 1$. From (3.2.8), (3.2.9) and (3.2.10),

$$p_{i(i-2)} = \sum_{m=1}^{i-2} b_m \quad , \ i = 3,4,\ldots,n \tag{3.2.11}$$

Consider now the fourth subdiagonal of $\underline{P}$. The elements along this diagonal, $p_{i(i-4)}$, $i = 5,6,\ldots,n$ are given by:

$$p_{i(i-4)} = \frac{R_{(n-i+4)3}}{R_{(n-i+4)1}}, \quad (i = 5,6,\ldots,n) \tag{3.2.12}$$

Using the relation between the Routh array elements,

$$R_{ij} = R_{(i-2)(j+1)} - \frac{R_{(i-2)1}\, R_{(i-1)\,(j+1)}}{R_{(i-1)1}} \qquad (3.2.13)$$

it is very simple to show, from (3.2.9) and (3.2.10), that

$$\left.\begin{aligned} p_{51} &= b_3\, p_{31} \\ p_{i(i-4)} &= p_{(i-2)(i-4)}\, b_{(i-4)} + p_{(i-1)(i-5)} \\ & \qquad (i=6,7,\ldots,n) \end{aligned}\right\} \qquad (3.2.14)$$

This procedure can be repeated and in general

$$p_{ij} = \left\{\begin{array}{lll} 0 & j > i & \\ 0 & i+j \text{ odd} & \\ 1 & i=j & \\ p_{(i-2)1}\, b_{(i-2)} & ,\ j=1\ ,\ i \neq j & \\ p_{(i-2)j}\, b_{(i-2)} + p_{(i-1)(j-1)} & & i+j \text{ even} \\ \qquad\qquad .\ i \neq j\ ,\ j \neq 1 & & \\ \qquad (i,j = 1,2,\ldots,n) & & \end{array}\right\} \qquad (3.2.15)$$

In words, any element p_{ij} is found by multiplying the element situated in the same column and $(i-2)$-th row by b_{i-2} and adding it to the element situated in $(i-1)$-th row and $(j-1)$-th column. Of course, the first row is to be first filled in as $(1, 0, 0, \ldots, 0, 0)$.

It may be similarly shown that the elements q_{ij} of $\underline{P}^{-1}$ are given as:

$$q_{ij} = \left\{ \begin{array}{ll} 0 & j > i \\ 0 & j + i \text{ odd} \\ 1 & i = j \\ -b_1 \, q_{(i-1)2}\,, \quad i \neq j\,, \quad j = 1 & \\ q_{(i-1)(j-1)} + (-b_j)\, q_{(i-1)(j+1)} & \\ \qquad i \neq j\,, \quad j \neq 1 & \\ \qquad (i,j = 1,2,\ldots,n) & \end{array} \right\} \begin{array}{l} \\ \\ \\ i+j \\ \text{even} \end{array} \qquad (3.2.16)$$

In words, first write down the first row of $\underline{P}^{-1}$ as $(1,0,0,\ldots,0,0)$. Then any element q_{ij} is found by multiplying the element in the $(i-1)$-th row and $(j+1)$-th column by $(-b_j)$ and adding it to the element in the $(i-1)$-th row and the $(j-1)$-th column. Power [20] has suggested a flow graph rule to compute $\underline{P}^{-1}$.

It is quite interesting to observe the following points. First, if $\underline{P}_n(\underline{P}_n^{-1})$ is known as in (3.2.15) [(3.2.16)], then $\underline{P}_k(\underline{P}_k^{-1})$ where $k < n$ is the descending principal minor of order k of $\underline{P}_n$ $(\underline{P}_n^{-1})$.

Second, if $\underline{P}$ is given as in (3.2.17) where p_{ij}'s are to be expressed in terms of b_i's as in (3.2.15), then the elements of the second subdiagonal of $\underline{P}^{-1}$, from the upper left corner to the lower right corner, are:

$$-p_{31}\,, \quad -p_{42}\,, \quad -p_{53}\,, \; \ldots\,, \; -p_{n(n-2)}$$

The fourth subdiagonal elements, from the upper left corner to the lower right corner, are given by the 2 x 2 determinants

$$\begin{vmatrix} p_{31} & 1 \\ p_{51} & p_{53} \end{vmatrix}, \begin{vmatrix} p_{42} & 1 \\ p_{62} & p_{64} \end{vmatrix}, \begin{vmatrix} p_{53} & 1 \\ p_{73} & p_{75} \end{vmatrix}, \ldots, \begin{vmatrix} p_{(n-2)(n-4)} & 1 \\ p_{n(n-4)} & p_{n(n-2)} \end{vmatrix}$$

as illustrated in (3.2.17).

$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ p_{31} & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_{42} & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ p_{51} & 0 & p_{53} & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & p_{62} & 0 & p_{64} & 0 & 1 & \cdots & 0 & 0 \\ p_{71} & 0 & p_{73} & 0 & p_{75} & 0 & \cdots & 0 & 0 \\ 0 & p_{82} & 0 & p_{84} & 0 & p_{86} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 & 1 \end{bmatrix} \quad \ldots \quad (3.2.17)$$

These results are exactly analogous to the previous ones except that now everything starts from the upper left corner. This is because of the way in which the b_i's and the R_{ij}'s are related $\left[b_n = R_{21}, \ b_{n-1} = R_{31}, \ b_{n-2} = R_{41}/R_{21}, \ldots, \ b_1 = R_{(n+1)1}/R_{(n-1)1} \right]$.

Finally, it is observed that only the first column of the Routh array is used to compute the b_i's from (3.2.10) as against the previous method which uses all the columns of the Routh array. Equations (3.2.15) and

(3.2.16) are exceedingly simple to remember and use in order to determine $\underline{P}$ and $\underline{P}^{-1}$ respectively. Number of multiplications involved in the computation is considerably reduced. It is for these reasons, equations (3.2.15) and (3.2.16) are recommended for computing $\underline{P}$ and $\underline{P}^{-1}$ respectively.

One more method to compute the transformation matrices is available [20] . After computing the elements b_i's of $\underline{B}$ (from the Routh array), apply Power's transformation [20] to $\underline{B}$ so as to transform $\underline{B}$ to its phase-variable form. Thus

$$\underline{C} = \underline{M}\,\underline{B}\,\underline{M}^{-1} \tag{3.2.18}$$

where $\underline{M}$ can be found by Power's method. Therefore,

$$\therefore \quad \underline{P}^{-1} = \underline{M}$$

and (3.2.19)

$$\underline{P} = \underline{M}^{-1}$$

As this method is not superior to the one discussed above [equations (3.2.15) and (3.2.16)], detailed presentation of this method is not aimed at. However, further details may be obtained from [20] .

3.3 TRANSFORMATION FROM ARBITRARY FORM TO SCHWARZ FORM

Recently there have been many papers [19] discussing the transformation of a controllable system whose description is in arbitrary form to the phase-variable form. This section, following Tuel's [22] approach, derives a

transformation which takes a controllable system in arbitrary form to one in Schwarz canonical form [21]. The transformation is extended to unforced systems, thereby pointing out the difference between forced and unforced systems from the viewpoint of such a transformation. A particular case wherein the given system has the phase-variable form description is considered and under a certain condition, the corresponding transformation is shown to be the one discussed in the last section.

Consider a controllable system

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{b}\ u \tag{3.3.1}$$

where $\underline{A}$ is some arbitrary $n \times n$ matrix. A transformation

$$\underline{x} = \underline{H}\ \underline{y} \tag{3.3.2}$$

is sought such that (3.3.1) becomes

$$\dot{\underline{y}} = \underline{B}\ \underline{y} + \underline{f}\ u \tag{3.3.3}$$

where $\underline{B}$ is in Schwarz form and

$$\underline{f} = (\ 0,\ 0,\ .\ .\ .\ ,\ 0,\ 1\ )^T \tag{3.3.4}$$

From (3.3.1), (3.3.2) and (3.3.3) it is seen that

$$\underline{H}\,\underline{B} = \underline{A}\,\underline{H} \tag{3.3.5}$$

and

$$\underline{H}\,\underline{f} = \underline{b} \tag{3.3.6}$$

Let

$$\underline{H} = \left[\ \underline{h}_1 \mid \underline{h}_2 \mid \cdots \mid \underline{h}_n \right] \tag{3.3.7}$$

Then (3.3.5) yields

$$\left[\underline{A}\,\underline{h}_1 \,\middle|\, \underline{A}\,\underline{h}_2 \,\middle|\, \cdots \,\middle|\, \underline{A}\,\underline{h}_n\right] = \left[\underline{h}_1 \,\middle|\, \underline{h}_2 \,\middle|\, \cdots \,\middle|\, \underline{h}_n\right] \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ -b_1 & 0 & 1 & . & . & . & 0 & 0 \\ 0 & -b_2 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & 0 & . & . & . & -b_{n-1} & -b_n \end{bmatrix} \quad \ldots \quad (3.3.8)$$

i.e.,

$$\left[\underline{A}\,\underline{h}_1 \,\middle|\, \underline{A}\,\underline{h}_2 \,\middle|\, \cdots \,\middle|\, \underline{A}\,\underline{h}_n\right] = \left[-b_1\,\underline{h}_2 \,\middle|\, \underline{h}_1 - b_2\,\underline{h}_3 \,\middle|\, \cdots \,\middle|\, \underline{h}_{n-2} - b_{n-1}\,\underline{h}_n \,\middle|\, \underline{h}_{n-1} - b_n\,\underline{h}_n\right] \quad \ldots \quad (3.3.9)$$

From (3.3.6) it is clear that

$$\underline{h}_n = \underline{b} \qquad (3.3.10)$$

Equations (3.3.9) and (3.3.10) give the following results:

$$\left.\begin{aligned} \underline{h}_n &= \underline{b} \\ \underline{h}_{n-1} &= \underline{A}\,\underline{b} + b_n\,\underline{b} \\ \underline{h}_{n-k} &= \underline{A}\,\underline{h}_{n-k+1} + b_{n-k+1}\,\underline{h}_{n-k+2} \end{aligned}\right\} (3.3.11)$$

$$(k=2,3,\ldots,n-1)$$

The controllability condition imposed on the pair $(\underline{A}, \underline{b})$ implies that the composite matrix

$$\underline{Q} = \left[\underline{b} \,\middle|\, \underline{A}\,\underline{b} \,\middle|\, \underline{A}^2\,\underline{b} \,\middle|\, \cdots \,\middle|\, \underline{A}^{n-1}\,\underline{b}\right] \qquad (3.3.12)$$

is nonsingular. From (3.3.7) and (3.3.11), $\underline{H}$ is written as:

$$\underline{H} = \left[\underline{A}^{n-1}\,\underline{b} + b_n\,\underline{A}^{n-2}\,\underline{b} + \dots \,\vdots\, \dots \,\vdots\, \underline{A}^2\underline{b} + b_n\underline{A}\,\underline{b} + b_{n-1}\,\underline{b} \,\vdots\, \underline{A}\,\underline{b} + b_n\underline{b} \,\vdots\, \underline{b} \right] \quad \dots \quad (3.3.13)$$

By performing elementary column operations on $\underline{H}$, the equivalence between $\underline{H}$ and $\underline{Q}$ can be established. Hence $\underline{H}$ is nonsingular if and only if the aforementioned controllability condition is satisfied.

It is interesting to note that the knowledge of the b_i's , $i = 1,2,\dots,n$, the elements of the Schwarz matrix $\underline{B}$, is required to determine $\underline{H}$ in (3.3.13). Tuel's transformation matrix [22] $\underline{K}$ ($\underline{K}$ transforms the system (3.3.1) to the one given by (3.3.3) where now $\underline{B}$ is in phase-variable form) also involves the elements of the characteristic equation of $\underline{A}$. An alternative procedure to construct $\underline{K}$ without apriori knowledge of the coefficients of the characteristic equation of $\underline{A}$ is given by the Luenberger [15] . Such a procedure does not seem possible in the case of $\underline{H}$ matrix because of the fact that the b_i's in $\underline{B}$ matrix are distributed throughout its rows and columns.

From the main assumption, the system (3.3.3) is controllable. This fact leads to the following conclusion: A sufficient condition for a system which is in Schwarz form to be controllable is that the system input vector must have a 1 in the n-th position and zeros everywhere else.

Consider now a free system

$$\dot{\underline{x}} = \underline{A}\ \underline{x} \tag{3.3.14}$$

Seek a transformation

$$\underline{x} = \underline{H}\ \underline{y} \tag{3.3.15}$$

such that

$$\dot{\underline{y}} = \underline{B}\ \underline{y} \tag{3.3.16}$$

$\underline{H}$ in (3.3.15) can now be determined as follows: Choose a vector $\underline{b}$ such that the pair $(\underline{A}, \underline{b})$ is controllable. Then form $\underline{H}$ as given in (3.3.7) and (3.3.11) with this $\underline{b}$.

Considering the forced system, once the forcing vector $\underline{b}$ is specified, the transformation discussed in this section is possible if and only if the pair $(\underline{A}, \underline{b})$ is controllable. But for free systems $\underline{b}$ can be so chosen as to ensure the controllability of the pair $(\underline{A}, \underline{b})$. In fact, there may be many transformation matrices $\underline{H}$ transforming (3.3.14) to (3.3.16).

Suppose $\underline{A}$ in the system (3.3.14) is in phase-variable form (denoted by $\underline{C}$). In section 2, a transformation

$$\underline{y} = \underline{P}\ \underline{x} \tag{3.3.17}$$

such that

$$\underline{P}\ \underline{C}\ \underline{P}^{-1} = \underline{B} \tag{3.3.18}$$

was discussed. The elements of $\underline{P}$ in (3.3.17) are determined from the knowledge of the elements of the Routh array of (3.3.14). Whereas the choice of $\underline{P}$ is unique, that of $\underline{H}$ is not so owing to the freedom available in choosing $\underline{b}$. It is intuitively felt that a particular choice

of $\underline{b}$ may make $\underline{P}^{-1}$ to be identically equal to $\underline{H}$. It can be established that when $\underline{b} = \underline{f}$, $\underline{P}^{-1} = \underline{H}$. It is quite simple to show that the system (3.3.1) with $\underline{A} = \underline{C}$ and $\underline{b} = \underline{f}$ is controllable. Determination of $\underline{H}$ with $\underline{A} = \underline{C}$ and $\underline{b} = \underline{f}$ can be accomplished from the knowledge of either the Routh array elements or the b_i's (See section 3.2). This can be used to determine $\underline{P}^{-1}(=\underline{H})$ in terms of the elements of the Routh array. In fact, this is how the author obtained the result in (3.2.7).

An example will now be considered.

Example:

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -12 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The characteristic equation for this is

$$\lambda^4 + 4\lambda^3 + 6\lambda^2 + 12\lambda + 3 = 0$$

Routh array is then formed.

$$\begin{array}{ccc} 1 & 6 & 3 \\ 4 & 12 & \\ 3 & 3 & \\ 8 & & \\ 3 & & \end{array}$$

Using the relationship between the Routh array elements

$(R_{ij}$'s) and the Schwarz elements (b_i's)

$$
\begin{aligned}
b_4 &= R_{21} &&= 4 \\
b_3 &= R_{31} &&= 3 \\
b_2 &= R_{41}/R_{21} &&= 2 \\
b_1 &= R_{51}/R_{31} &&= 1
\end{aligned}
$$

From (3.2.15)

$$
\underline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_1 & 0 & 1 & 0 \\ 0 & (b_1+b_2) & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}
$$

From (3.2.16)

$$
\underline{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -b_1 & 0 & 1 & 0 \\ 0 & -(b_1+b_2) & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}
$$

To compute $\underline{H}$, let $\underline{b} = \underline{f}$. Thus, using (3.3.11)

$$
\begin{aligned}
\underline{h}_4 &= \underline{f} &&= (\ 0\ 0\ 0\ 1\ )^T \\
\underline{h}_3 &= \underline{A}\ \underline{f} + b_4\ \underline{f} &&= (\ 0\ 0\ 1\ 0\ )^T \\
\underline{h}_2 &= \underline{A}\ \underline{h}_3 + b_3 \underline{h}_4 &&= (\ 0\ 1\ 0\ -3)^T \\
\underline{h}_1 &= \underline{A}\ \underline{h}_2 + b_2 \underline{h}_3 &&= (\ 1\ 0\ -1\ 0)^T
\end{aligned}
$$

Hence

$$
\underline{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}
$$

Observe that $\underline{H} = \underline{P}^{-1}$

3.4 TRANSFORMATION FROM SCHWARZ TO JORDAN FORM [23]

This section presents a transformation which takes a system from the Schwarz form to the Jordan form when the system has (i) distinct eigenvalues, (ii) repeated eigenvalues having simple degeneracy.

Consider a system described by

$$\dot{\underline{x}} = \underline{B}\,\underline{x} \tag{3.4.1}$$

It is desired to find a transformation

$$\underline{x} = \underline{T}\,\underline{z} \tag{3.4.2}$$

such that

$$\dot{\underline{z}} = \underline{J}\,\underline{z}$$

where $\underline{J} = \underline{T}^{-1}\,\underline{B}\,\underline{T}$ is the Jordan canonical form corresponding to the Schwarz form $\underline{B}$. Referring to Figure (3.1.1) it can be readily seen that

$$\underline{T} = \underline{P}\,\underline{V} \tag{3.4.3}$$

In section 3.2, it was stated that the elements of $\underline{P}$ can be easily determined in terms of b_i's by a simple rule [See equation (3.2.15)]. However, it is also possible to express the elements of $\underline{P}$ in terms of the b_i's explicitly [Note that relations in (3.2.15) are algorithmic in nature].

$$\left.\begin{array}{l} \underline{P} = \left[p_{ij} \right] \\ p_{ij} = \begin{cases} 0 & j > i \\ 0 & i + j \quad \text{odd} \\ \left.\begin{array}{ll} 1 & i = j \\ \sum\limits_{k=1}^{j} \sum\limits_{\ell=k+2}^{j+2} \cdots \sum^{i-2} (b_k \, b_\ell \ldots) & i \neq j \end{array}\right\} & i+j \text{ even} \end{cases} \\ \qquad (i,\ j = 1,\ 2,\ 3,\ \ldots,\ n) \end{array}\right\} \qquad (3.4.4)$$

This simplification helps in expressing $\underline{T}$ solely in terms of the elements of $\underline{B}$ and its eigenvalues.

<u>Case I</u>: <u>Distinct Eigenvalues</u>

If $\underline{T}$ is represented as

$$\underline{T} = \left[t_{ij} \right] , \qquad (3.4.5)$$

then by performing the matrix multiplication $\underline{T} = \underline{P}\ \underline{V}$, it can be shown, in the case of distinct eigenvalues, that

$$t_{ij} = \lambda_j^{i-1} + \left[\sum_{k=1}^{i-2} b_k \right] \lambda_j^{i-3} + \left[\sum_{k=1}^{i-4} \sum_{\ell=k+2}^{i-2} b_k b_\ell \right] \lambda_j^{i-5}$$

$$+ \left[\sum_{k=1}^{i-6} \sum_{\ell=k+2}^{i-4} \sum_{m=\ell+2}^{i-2} b_k b_\ell b_m \right] \lambda_j^{i-7} + \cdots .$$

$$(i,j = 1,2,\ldots,n) \qquad (3.4.6)$$

where $b_k = 0$ if $k < 0$ and λ_i denotes the i-th eigenvalue of the system in (3.4.1). An algorithm for

generating the t_{ij}'s is given below:

$$\left.\begin{aligned} t_{1k} &= 1, \ (k = 1,2,\ldots,n) \\ t_{2k} &= \lambda_k \ , \ (k = 1,2,\ldots,n) \\ t_{ik} &= \lambda_k \, t_{(i-1)k} + b_{i-2} \, t_{(i-2)k} \\ & \qquad (i = 3,4,\ldots,n) \\ & \qquad (k = 1,2,\ldots,n) \end{aligned}\right\} \qquad (3.4.7)$$

Typically in the case of a fifth order system, if

$$\underline{T} = \left[\underline{t}_1 \,\vdots\, \underline{t}_2 \,\vdots\, \underline{t}_3 \,\vdots\, \underline{t}_4 \,\vdots\, \underline{t}_5 \right]$$

and λ_k $(k = 1,2,\ldots,5)$ are the eigenvalues, then

$$\underline{t}_k = \begin{bmatrix} 1 \\ \lambda_k \\ \lambda_k^2 + b_1 \\ \lambda_k^3 + (b_1 + b_2)\lambda_k \\ \lambda_k^4 + (b_1 + b_2 + b_3)\lambda_k^2 + b_1 b_3 \end{bmatrix}$$

$$(k = 1,2,\ldots,5)$$

<u>Case II</u>: <u>Repeated Eigenvalues with Simple Degeneracy</u>

For simplicity, let us assume that $\underline{B}$ has one repeated eigenvalue λ_1 with multiplicity m, other eigenvalues being distinct. Let

$$\underline{T} = \left[\underline{t}_1 \,\vdots\, \underline{t}_2 \,\vdots\, \cdots \,\vdots\, \underline{t}_m \,\vdots\, \underline{t}_{m+1} \,\vdots\, \cdots \,\vdots\, \underline{t}_n \right] \qquad (3.4.8)$$

In forming the matrix product $\underline{T} = \underline{P}\,\underline{V}$, $\underline{V}$ is now the generalized Vandermonde matrix [20]. The resulting

transformation matrix $\underline{T}$ is such that the columns $\underline{t}_1$, $\underline{t}_{m+1}$, $\underline{t}_{m+2}$, ..., $\underline{t}_n$ can be computed using (3.4.7) and

$$\underline{t}_i = \frac{1}{(i-1)!} \frac{d^{i-1}}{d\lambda_1^{i-1}} \underline{t}_1 \quad (i = 2,3,\ldots,m) \tag{3.4.9}$$

The same procedure is valid even when $\underline{B}$ has more than one repeated eigenvalue, provided each eigenvalue has simple degeneracy.

<u>Example</u>: Consider the system (3.4.1) with

$$\underline{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix}$$

Here $b_1 = -1$, $b_2 = 0$, $b_3 = 2$ and $b_4 = 3$. $\underline{B}$ has eigenvalues -1, -1, 1 and -2. It can be checked that the eigenvalue -1 which has a multiplicity $m = 2$ has a simple degeneracy. By using the procedure outlined above, $\underline{T}$ can be computed as:

$$\underline{T} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 3 \\ 0 & 2 & 0 & -6 \end{bmatrix}$$

Now it can be easily verified that $\underline{T}^{-1}\,\underline{B}\,\underline{T} = \underline{J}$ where

$$\underline{J} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

It may be noted that the transformation $\underline{T}$ given above has some resemblance to the Vandermonde matrix.

3.5 SCHWARZ CANONICAL FORM FOR MULTIVARIABLE SYSTEMS

In this section, Schwarz canonical form for multivariable systems is suggested. A procedure for transforming the system description from arbitrary form to the Schwarz form is outlined. An example system is considered for illustration.

Luenberger [15] has recently proposed a transformation which takes a controllable multivariable system in arbitrary form to a set of coupled subsystems each of which being in phase-variable form. In this section, the transformation ($\underline{P}$) discussed in section 3.2 is applied to each such subsystem to bring it to the Schwarz form. Combining these two transformation processes, a single transformation may be obtained; this, therefore, takes a controllable multivariable system to a set of coupled subsystems each of which being in Schwarz form.

Let the multivariable system be given by

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{F}\,\underline{u} \qquad (3.5.1)$$

where $\underline{A}$ is in arbitrary form. Apply Luenberger's transformation [15]

$$\tilde{\underline{x}} = \underline{S}\,\underline{x} \qquad (3.5.2)$$

and the transformed system becomes

$$\dot{\tilde{\underline{x}}} = \tilde{\underline{A}}\,\tilde{\underline{x}} + \tilde{\underline{F}}\,\underline{u} \qquad (3.5.3)$$

The control input $\underline{u}$ can also be transformed as

$$\underline{v} \doteq \underline{G}\ \underline{u} \tag{3.5.4}$$

where $\underline{G}$ is an upper triangular matrix so that

$$\acute{\underline{F}}\ \underline{u} = \hat{\underline{F}}\ \underline{v} \tag{3.5.5}$$

where

$$\hat{\underline{F}} = \left[\begin{array}{ccc|c} \begin{matrix}0\\0\\ \vdots\\ 1\end{matrix} & & \underline{0} & \\ \underline{0} & \begin{matrix}0\\0\\ \vdots\\ 0\\ 1\end{matrix} & \underline{0} & \\ \hline & \underline{0} & & \begin{matrix}0\\0\\ \vdots\\ 0\\ 1\end{matrix} \end{array}\right] \tag{3.5.6}$$

Suppose (3.5.3) has m coupled subsystems corresponding to the submatrices $\tilde{\underline{A}}_1$, $\tilde{\underline{A}}_2$, ..., $\tilde{\underline{A}}_m$ of $\tilde{\underline{A}}$. Each of these submatrices $\tilde{\underline{A}}_k$, $k = 1,2,\ldots,m$ is in phase-variable form. Let $\underline{P}_1$, $\underline{P}_2$, ..., $\underline{P}_m$ be the transformation matrices such that $\underline{P}_k \tilde{\underline{A}}_k \underline{P}_k^{-1} = \underline{B}_k$, $k = 1,2,\ldots,m$ where $\underline{B}_k$ is in Schwarz form. Knowing $\tilde{\underline{A}}_k$, $\underline{P}_k$ can be computed as discussed in section 3.2. Now, form a composite transformation matrix $\underline{P}$ as

$$\underline{P} = \begin{bmatrix} \underline{P}_1 & & & \underline{0} \\ & \underline{P}_2 & & \\ & & \ddots & \\ \underline{0} & & & \underline{P}_m \end{bmatrix} \tag{3.5.7}$$

Clearly

$$\underline{y} = \underline{P}\,\underline{S}\,\underline{x} = \hat{\underline{P}}\,\underline{x} \text{ (say)} \tag{3.5.8}$$

transforms (3.5.1) to

$$\dot{\underline{y}} = \underline{B}\,\underline{y} + \hat{\underline{F}}\,\underline{v} \tag{3.5.9}$$

Note that in (3.5.9), the fact that $\underline{P}\,\hat{\underline{F}} = \hat{\underline{F}}$ has been used. The multivariable system in (3.5.9) is said to be in Schwarz canonical form. In (3.5.9), $\underline{B}$ has m submatrices $\underline{B}_k$, $k = 1,2,\ldots,m$ and each of these submatrices is in Schwarz form. Observe also that (3.5.9) still has coupling terms between subsystems though they might have been modified because of the transformation (3.5.8) between (3.5.3) and (3.5.9).

It is interesting to ask as to whether $\hat{\underline{P}}$ in (3.5.8) can be directly determined (i.e. without having to compute $\tilde{\underline{A}}$). It seems to the author that it is not possible for (i) the dimensions of the submatrices $\tilde{\underline{A}}_k$ or $\underline{B}_k$ are not known apriori (i.e. before computing $\tilde{\underline{A}}_k$) and (ii) $\hat{\underline{P}}$ does require the knowledge of the Routh array of the submatrices.

<u>Example</u>: Consider the system in (3.5.1) with

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix} \text{ and } \underline{F} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1/5 \\ 0 & 1/6 \end{bmatrix}$$

Luenberger's transformation matrix $\underline{S}$ can be found as:

$$\underline{S} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{6} & 1 & -\frac{3}{2} & 0 & 0 & 0 \\ -\frac{1}{6} & -2 & -\frac{9}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -5 & 3 \\ 0 & 0 & 0 & -2 & 25 & -18 \\ 0 & 0 & 0 & 8 & -125 & 108 \end{bmatrix}$$

The matrix triple product $\underline{S}\ \underline{A}\ \underline{S}^{-1}$ results in

$$\underline{\hat{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -6 & -11 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -120 & -74 & -15 \end{bmatrix}$$

Also

$$\tilde{\underline{F}} = \underline{S}\ \underline{F} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that in this simple example, $\underline{A}$ does not have any coupling terms and $\tilde{\underline{F}} = \hat{\underline{F}}$ (i.e. $\underline{G} = \underline{I}$). Clearly

$$\tilde{\underline{A}}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

and

$$\tilde{\underline{A}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -120 & -74 & -15 \end{bmatrix}$$

Now $\tilde{\underline{A}}_1$ and $\tilde{\underline{A}}_2$ can be transformed to their Schwarz forms $\underline{B}_1$ and $\underline{B}_2$ respectively and the corresponding transformation matrices $\underline{P}_1$ and $\underline{P}_2$ can be determined. The composite transformation matrix $\underline{P}$ is obtained as

$$\underline{P} = \left[\begin{array}{c:c} \underline{P}_1 & \underline{0} \\ \hdashline \underline{0} & \underline{P}_2 \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 & 0 & 1 \end{array}\right]$$

The transformation matrix $\underline{\hat{P}} = \underline{P}\,\underline{S}$ is computed next.

$$\underline{\hat{P}} = \begin{bmatrix} 1/6 & -1/2 & 1/2 & 0 & 0 & 0 \\ -1/6 & 1 & -3/2 & 0 & 0 & 0 \\ 0 & -5/2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & -5 & 3 \\ 0 & 0 & 0 & -2 & 25 & -18 \\ 0 & 0 & 0 & 12 & -165 & 132 \end{bmatrix}$$

Thus $\underline{y} = \underline{\hat{P}}\,\underline{x}$ transforms the system (3.5.1) to (3.5.9)

where $\underline{\hat{F}} = \underline{\hat{F}}'$ (computed above) and

$$\underline{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -10 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -8 & 0 & 1 \\ 0 & 0 & 0 & 0 & -66 & -15 \end{bmatrix}$$

3.6 CONCLUSION

Various transformations related to the Schwarz canonical form have been quite extensively discussed in this chapter. Also in the last chapter, transformations between the three Schwarz forms were dealt with. However, a gap still remains to be filled in and that is, finding the inverse of the transformation matrix $\underline{T}$ (between $\underline{B}$ and $\underline{J}$). Analogous to the result available for the inverse of the Vandermonde matrix [24], it may be possible to device a method of finding $\underline{T}^{-1}$. Also some practical use may be found for Schwarz form for multivariable systems.

CHAPTER - IV

NETWORK INTERPRETATION FOR SCHWARZ CANONICAL FORM

4.1 INTRODUCTION

In chapter II, the Schwarz canonical matrix was developed through the use of continued fractions. Also, the Schwarz matrix is tridiagonal. So it is reasonable to expect a ladder network interpretation to the Schwarz canonical form. In this chapter, the three forms of the Schwarz matrix $\underline{S}_1$, $\underline{S}_2$ and $\underline{S}_3$ introduced in chapter II are associated with resistively terminated L-C ladder networks. Section 2 derives elaborately such a network interpretation. As a byproduct, simple transformations between $\underline{S}_1$ and $\underline{S}_2$ and between $\underline{S}_1$ and $\underline{S}_3$ are obtained. The possibilities of using such an interpretation for both analysis and synthesis of systems are also investigated in the same section. Section 3 extends this network interpretation to multivariable systems in Schwarz canonical form. An example is also contained in this section.

4.2 NETWORK INTERPRETATION TO SINGLE-VARIABLE SYSTEMS IN SCHWARZ FORM

In this section, it is shown that the Schwarz matrices $\underline{S}_1$, $\underline{S}_2$ and $\underline{S}_3$ are associated with resistively terminated L-C ladder networks. As a byproduct of such a network interpretation, the transformations between $\underline{S}_1$,

$\underline{S}_2$ and $\underline{S}_3$ are derived in a simple way.

Suppose the transfer function of a system is of the form:

$$G(s) = \frac{k}{a_o s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \tag{4.2.1}$$

By a proper normalization procedure [See Appendix II, equation (A-2-8)], k, a_n and a_o can be made equal to unity. The normalized transfer function, say $F(s)$, is given by:

$$F(s) = \frac{1}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + 1} \tag{4.2.2}$$

It is quite interesting to observe that the transfer function (4.2.2) can be modeled by means of a resistively terminated L-C ladder network [25]. The procedure to arrive at such a model will now be explained elaborately, following the references [25,26,27].

The model sought is of the form as shown in Figure (4.2.1-a). Using the same notation as in the reference [26], the transfer admittance function $Y_{12}(s)$ and transfer function $G_{12}(s)$ for the model are written as:

$$-Y_{12}(s) = \frac{I_2(s)}{V_1(s)} = \frac{y_{12}(s)\ G_2}{G_2 + y_{22}(s)} \tag{4.2.3}$$

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)} = -\frac{y_{12}(s)}{G_2+y_{22}(s)} \qquad (4.2.4)$$

where $G_2 = 1/R_2$ and $y_{12}(s)$ and $y_{22}(s)$ are respectively the short circuit transfer and the short circuit driving-point admittances of the network shown by a rectangular block in Figure (4.2.1-a). If $G_2 = 1$, then $G_{12} = Y_{12}$. Then following Cauer's method (Cauer's first form), the elements of the model can be determined quite easily. If

$$F(s) = \frac{1}{P(s)} = \frac{1}{p_1(s) + p_2(s)} \qquad (4.2.5)$$

where

$$P(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + 1 \qquad (4.2.6)$$

and $p_1(s)$ and $p_2(s)$ are even and odd parts of $P(s)$ respectively. Rewriting (4.2.5) as

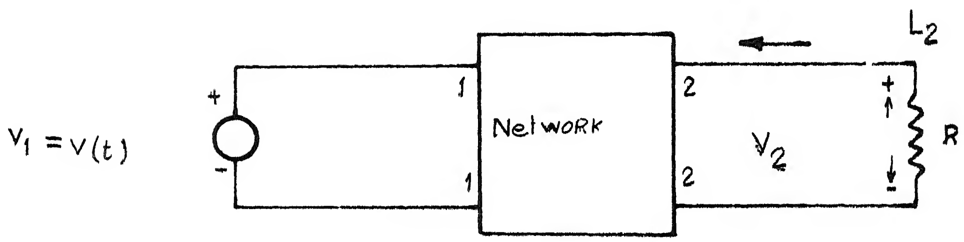
$$F(s) = \frac{1/p_1(s)}{1 + p_2(s)/p_1(s)} \qquad (4.2.7)$$

and comparing (4.2.7) and (4.2.4) result in:

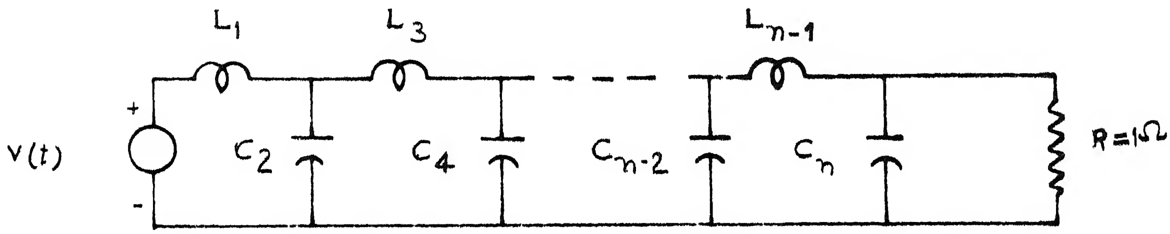
$$-y_{12}(s) = 1/p_1(s) \qquad (4.2.8)$$

$$y_{22}(s) = p_2(s)/p_1(s) \qquad (4.2.9)$$

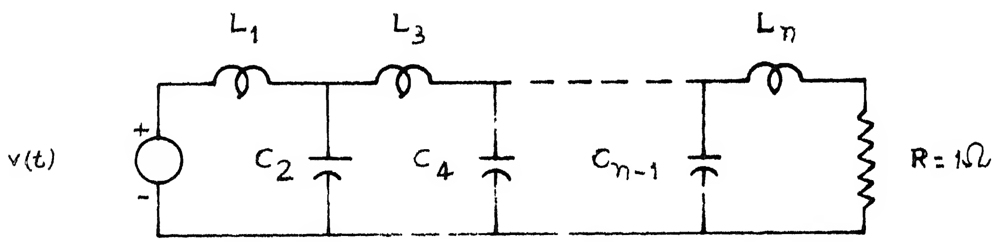
$y_{22}(s)$ in (4.2.9) can then be expanded in a continued fraction to determine the values of the inductors and capacitors of the network in Figure (4.2.1-a). The model



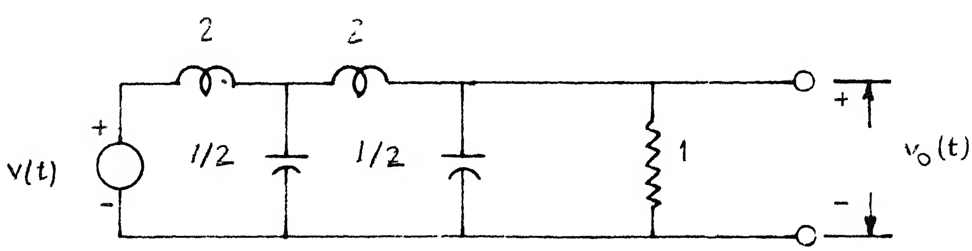
(a)



(b) n even



(c) n odd



Henrys, Farads, Ohms

(d)

NETWORK INTERPRETATION FOR SCHWARZ CANONICAL FORM FOR SINGLEVARLABLE SYSTEMS

with all its elements known can now be represented as in Figure (4.2.1-b) or as in Figure (4.2.1-c) depending upon whether n is even or odd.

Example: Consider

$$F(s) = \frac{v_o(s)}{v(s)} = \frac{1}{s^4 + 2s^3 + 3s^2 + 4s + 1}$$

From (4.2.9)

$$y_{22}(s) = \frac{p_1(s)}{p_2(s)} = \frac{s^4 + 3s^2 + 1}{2s^3 + 4s}$$

$$= \frac{1}{2}s + \cfrac{1}{2s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{2s}}}$$

Hence the proposed model for this transfer function is as in Figure (4.2.1-d).

As Routh's stability criterion may be formulated through continued fraction [16, 28] it is quite reasonable to expect that the elements of the ladder network can be determined from the knowledge of the Routh array of $P(s)$ in (4.2.6) [25] . Define $H(s)$ as

$$H(s) = \frac{a_1 s^{n-1} + a_3 s^{n-3} + \cdots}{s^n + a_2 s^{n-2} + \cdots} \tag{4.2.10}$$

$H(s)$ can be expanded in a continued fraction.

$$H(s) = \cfrac{1}{\cfrac{1}{a_1} s + \cfrac{1}{\cfrac{a_1}{b_1} s + \cfrac{1}{\cfrac{b_1}{c_1} s + \cfrac{1}{\cfrac{c_1}{d_1} s + \ddots}}}} \tag{4.2.11}$$

where

$$\begin{aligned} b_1 &= a_2 - \frac{a_3}{a_1} \\ c_1 &= a_3 - \frac{a_1 b_3}{b_1} \\ d_1 &= b_3 - \frac{b_1\ c_3}{c_1} \end{aligned} \tag{4.2.12}$$

...etc.

From (4.2.12) it is obvious that

$$\begin{aligned} b_1 &= R_{31} \\ c_1 &= R_{41} \\ d_1 &= R_{51} \end{aligned} \tag{4.2.13}$$

...etc.

where R_{ij}'s are the elements of the Routh array of $P(s)$. Define

$$D_k = \left\{ \begin{array}{ll} L_k \ , & k \text{ odd} \\ C_k \ , & k \text{ even} \end{array} \right\} \tag{4.2.14}$$

$(k = 1,2,\ldots,n)$

where L and C refer to the inductance and capacitance respectively. Then from (4.2.11) and (4.2.14), it is seen that

$$D_k = \frac{R_{(n-k+1)1}}{R_{(n-k+2)1}}, \quad (k = 1,2,\ldots,n) \tag{4.2.15}$$

Thus the first column of the Routh array can be effectively used to determine the elements of the ladder network. If the system is stable, all the D_k's $(k=1,2,\ldots,n)$ will be positive. In other words, stability is a necessary condition for obtaining a network [Figure (4.2.1-a)] which is physically realizable.

From (2.3.10) and (4.2.15), it is very simple to see that

$$\begin{aligned} b_i &= \frac{1}{D_i\, D_{i+1}}, \quad (i = 1,2,\ldots,n-1) \\ b_n &= \frac{1}{D_n} \end{aligned} \tag{4.2.16}$$

It will be next shown that a proper choice of state variables will yield the system matrix as any one of the three Schwarz canonical matrices ($\underline{S}_1$, $\underline{S}_2$ and $\underline{S}_3$) for the system which is modeled by a network as in Figure (4.2.1). Although in the following treatment n is assumed to be even, an exactly similar treatment is possible for 'n odd' case also.

Consider Figure (4.2.1-b). If charges associated with capacitors and flux-linkages associated with inductors

are chosen as state variables, the system equations can be written as:

$$\left.\begin{aligned}
\dot{\underline{s}} &= \underline{D}\,\underline{s} + \underline{e}\,v \\
\text{where}\quad \underline{s} &= (\psi_1, q_2, \psi_3, q_4, \ldots)^T \\
\underline{D} &= \begin{bmatrix} 0 & -1/C_2 & 0 & \cdots & 0 & 0 \\ 1/L_1 & 0 & -1/L_3 & \cdots & 0 & 0 \\ 0 & 1/C_2 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1/L_{n-1} & -1/C_n \end{bmatrix} \\
\text{and}\quad \underline{e} &= (1, 0, \ldots, 0)^T
\end{aligned}\right\} \tag{4.2.17}$$

Choose the state vector $\underline{w}$ such that

$$\left.\begin{aligned}
w_1 &= -\psi_1 \\
w_i &= \begin{cases} -\left(\prod_{k=1}^{i-1} -D_k\right)\psi_i & i \text{ odd} \\[2ex] -\left(\prod_{k=1}^{i-1} -D_k\right) q_i & i \text{ even} \end{cases} \\
&\qquad\qquad (i = 2,3,\ldots,n)
\end{aligned}\right\} \tag{4.2.18}$$

In other words, seek a transformation

$$\underline{w} = \underline{T}\,\underline{s} \tag{4.2.19}$$

where

$$\underline{T} = \text{diag.}\left\{-1,\ L_1,\ -L_1C_2,\ L_1C_2L_3, \ldots\right\} \tag{4.2.20}$$

It may be easily verified that

$$\underline{T}\ \underline{D}\ \underline{T}^{-1} = \underline{S}_2$$

and

$$\underline{T}\ \underline{c} = \underline{d} \tag{4.2.21}$$

where

$$\underline{d} = (-1,\ 0,\ \ldots,\ 0)^T$$

Therefore

$$\underline{\dot{w}} = \underline{S}_2\ \underline{w} + \underline{d}\ v \tag{4.2.22}$$

Thus when the state variables w_i $(i=1,2,\ldots,n)$ are chosen as in (4.2.18), the system equations will be given by (4.2.22).

Consider now the state variables y_i $(i=1,2,\ldots,n)$ given by

$$y_i = \begin{cases} \dfrac{q_i}{\left(\prod_{k=1}^{i} -D_k\right)}, & i \text{ even} \\[2ex] \dfrac{\psi_i}{\left(\prod_{k=1}^{i} -D_k\right)}, & i \text{ odd} \end{cases} \qquad (i=1,2,\ldots,n) \tag{4.2.23}$$

or

$$\underline{y} = \underline{U}\ \underline{s} \tag{4.2.24}$$

where

$$\underline{U} = \text{diag.} \left\{ \left(-\frac{1}{L_1}\right), \left(\frac{1}{L_1 C_2}\right), \left(-\frac{1}{L_1 C_2 L_3}\right), \ldots \right\} \qquad \ldots \qquad (4.2.25)$$

With (4.2.24), (4.2.17) becomes

$$\dot{\underline{y}} = \underline{B}\,\underline{y} + \underline{g}\,v \qquad (4.2.26)$$

where

$$\underline{g} = \left(-\frac{1}{L_1}, 0, \ldots, 0\right)^T \qquad (4.2.27)$$

Pick the state variables $r_i (i=1,2,\ldots,n)$ such that

$$r_{n-i+1} = \begin{cases} -\dfrac{1}{\prod_{k=1}^{i} D_k}\,\psi_i\,, & i \text{ odd} \\[2ex] -\dfrac{1}{\prod_{k=1}^{i} D_k}\,q_i\,, & i \text{ even} \end{cases} \qquad (i=1,2,\ldots,n) \qquad (4.2.28)$$

i.e.

$$\underline{r} = \underline{V}\,\underline{s} \qquad (4.2.29)$$

where

$$\underline{V} = \begin{bmatrix} & \underline{0} & & & -\dfrac{1}{(L_1 C_2 L_3 \ldots C_n)} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & -\dfrac{1}{L_1 C_2} & & & \\ -\dfrac{1}{L_1} & & & \underline{0} & \end{bmatrix} \qquad (4.2.30)$$

With (4.2.29), (4.2.17) is transformed to

$$\dot{\underline{r}} = \underline{S}_3\, \underline{r} + \underline{h}\, v \tag{4.2.31}$$

where

$$\underline{h} = (0,0,\ldots,0,-1/L_1)^T \tag{4.3.32}$$

Can this investigation help in getting the transformations between $\underline{S}_1$ and $\underline{S}_2$ and between $\underline{S}_1$ and $\underline{S}_3$? Let these transformation matrices be respectively $\underline{M}_2$ and $\underline{M}_3$. Clearly

$$\underline{y} = \underline{U}\, \underline{s} = \underline{U}\, \underline{T}^{-1}\, \underline{w} = \underline{M}_2\, \underline{w}$$

i.e.

$$\underline{M}_2 = \underline{U}\; \underline{T}^{-1} \tag{4.2.33}$$

As $\underline{U}$ and $\underline{T}$ are diagonal, $\underline{M}_2$ becomes diagonal. It is readily seen that

$$\left.\begin{aligned} (\underline{M}_2)_{11} &= 1/D_1 \\ (\underline{M}_2)_{ii} &= \frac{1}{\left(\prod_{k=1}^{i-1} D_k\right)^2 D_i}, \quad (i=2,3,\ldots,n) \end{aligned}\right\} \tag{4.2.34}$$

If it is required to know $(\underline{M}_2)_{ii}$'s $(i=1,2,\ldots,n)$ in terms of b_i's, then equations (4.2.16) may be solved to get:

$$D_i = \begin{cases} \dfrac{\prod\limits_{m=0,2,4,\ldots}^{n-i-1} (b_{n-m})^2}{\prod\limits_{k=0,1,2,\ldots}^{n-i} b_{n-k}}, & (n-i) \ \text{odd} \\[2em] \dfrac{\prod\limits_{m=1,3,5,\ldots}^{n-i-1} (b_{n-m})^2}{\prod\limits_{k=0,1,2,\ldots}^{n-i} b_{n-k}}, & (n-i) \ \text{even} \\ (i=1,2,\ldots,n-1) & \end{cases} \tag{4.2.35}$$

$$D_n = 1/b_n$$

Using (4.2.34) and (4.2.35), $(\underline{M}_2)_{ii}$ $(i=1,2,\ldots,n)$ can be computed from b_i's $(i=1,2,\ldots,n)$. The diagonal transformation $\underline{M}_2$ is simpler than the one ($\underline{T}_2$) obtained in chapter II See (2.2.16) . As

$$\underline{y} = \underline{U}\,\underline{s} = \underline{U}\,\underline{V}^{-1}\,\underline{r} , \tag{4.2.36}$$

$\underline{M}_3$ can be determined from

$$\underline{M}_3 = \underline{U}\ \underline{V}^{-1} \tag{4.2.37}$$

It is quite simple to see that $\underline{M}_3$ is a cross-diagonal matrix and that

$$\left.\begin{array}{lll} (\underline{M}_3)_{ij} & = 0 & \text{when } j \neq n-i+1 \\ (\underline{M}_3)_{i(n-i+1)} & = (-1)^{i+1} & \\ & & (i=1,2,\ldots,n) \end{array}\right\} \qquad (4.2.38)$$

Thus it has been shown that each of the mathematical models (4.2.22), (4.2.26) and (4.2.31) can be thought of representing the state equations of a ladder network in Figure (4.2.1), when the state variables are appropriately chosen. In fact, each state variable is proportional to a corresponding variable (charge or flux linkage) associated with an element (capacitor or inductor) of the ladder network, the proportionality constant being related to the parameters of the network.

It may be observed that the mathematical model (4.2.22) or (4.2.26) or (4.2.31) has been 'realized' by a physical network shown in Figure (4.2.1). In this sense, it is a synthesis problem. It may be expected that such an interpretation may facilitate analysis too. Consider any physical system which can be modeled by a ladder network in Figure (4.2.1). The state equations for such a system can be directly written in the form (4.2.22) or (4.2.26) or (4.2.31). If the system transfer function is given as in (4.2.1), then by employing the procedure outlined above, the state equations

Suppose the given system is described by

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{b}\ u \qquad (4.2.39)$$

where $\underline{A}$ is any arbitrary $n \times n$ matrix. Reference [21] has given a transformation (See also chapter III) which takes the system in (4.2.39) to the one given by:

$$\dot{\underline{y}} = \underline{B}\ \underline{y} + \underline{f}\ u \qquad (4.2.40)$$

where

$$\underline{f} = (\ 0,0,\ldots,0,1\ )^T \qquad (4.2.41)$$

provided (4.2.39) refers to a controllable system. Equation (4.2.40) is same as the equation (4.2.26) except for the input vector $\underline{f}$. As $\underline{f}$ is different from $\underline{g}$, the network corresponding to (4.2.40) is not the one in Figure (4.2.1) and it is given in Figure (4.2.2).

4.3 NETWORK INTERPRETATION FOR MULTIVARIABLE SYSTEMS IN SCHWARZ FORM

Schwarz canonical form for multivariable systems was introduced in chapter III. It has been shown that any controllable multivariable system

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{F}\ \underline{u} \qquad (4.3.1)$$

can be transformed through

$$\underline{y} = \underline{\hat{P}}\ \underline{x} \qquad (4.3.2)$$

to a system

$$\dot{\underline{y}} = \underline{B}\ \underline{y} + \underline{\hat{F}}\ \underline{v} \qquad (4.3.3)$$

The system in (4.3.3) consists of coupled subsystems each of which being in Schwarz form. Thus, each of these canonical subsystems can be associated with a resistively terminated L-C ladder network with voltage or current drivers at appropriate locations in the ladder network. To account for couplings between canonical subsystems, dependent drivers will appear in the networks. These ideas may be better explained with respect to an example.

<u>Example</u>: For convenience, consider a multivariable system (4.3.1) which is already in phase-variable form [i.e. $\underline{A} = \underline{\tilde{A}}$ and $\underline{F} = \underline{\tilde{F}}$. See section 3.5] with

$$\underline{\tilde{A}} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -3 & -1 & -4 & -2 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -6 & -9 & -2 \end{array}\right] \text{ and } \underline{\tilde{F}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 5 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right]$$

By the method discussed in chapter III, the transformation matrix $\hat{\underline{P}}$ is found to be

$$\hat{\underline{P}} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \end{array}\right]$$

The input transformation matrix $\underline{G}$ See (3.5.4) can be written as

$$\underline{G} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

The transformed system becomes

$$\dot{\underline{y}} = \underline{B}\,\underline{y} + \hat{\underline{F}}\,\underline{v} . \qquad (4.3.3)$$

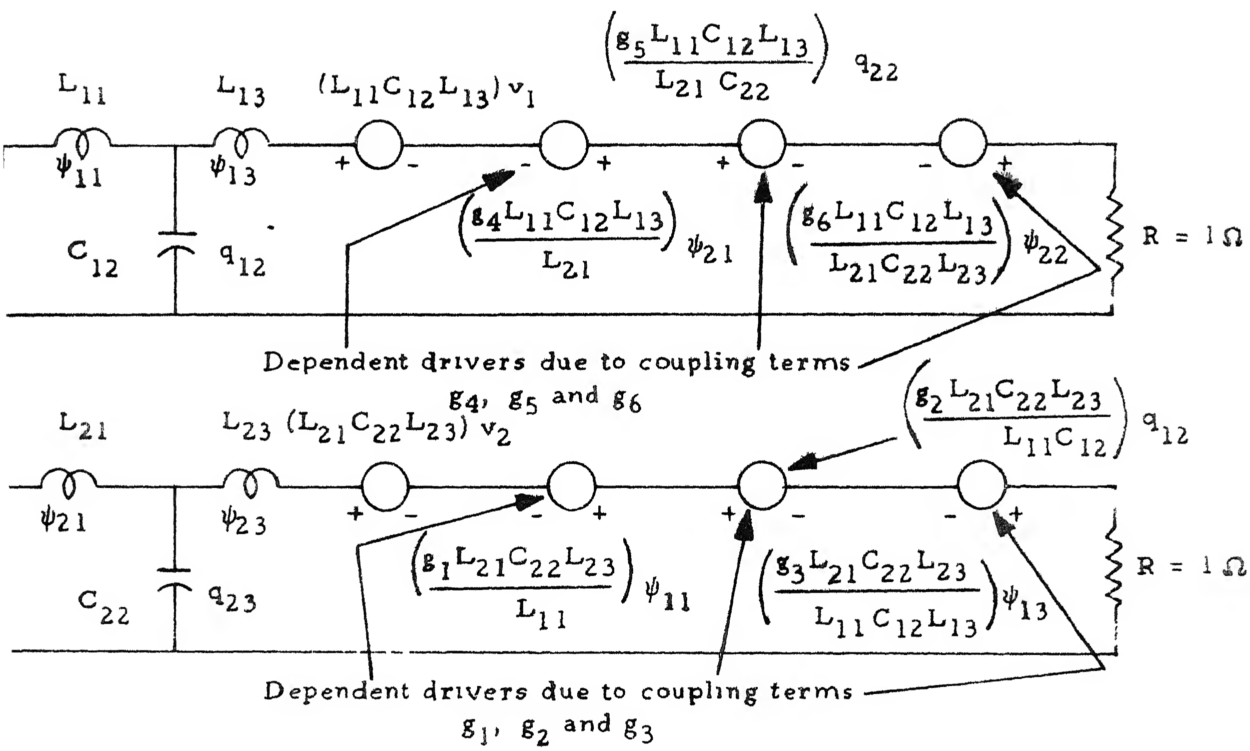
Where

$$\underline{B} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -2 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \\ 5 & -2 & -3 & 0 & -6 & -2 \end{array}\right] \text{ and } \hat{\underline{F}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right]$$

Note that the coupling terms in $\tilde{\underline{A}}$ and $\underline{B}$ are different (owing to the transformation). Write down $\underline{B}$ in the following form:

$$\underline{B} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -b_{11} & 0 & 1 & 0 & 0 & 0 \\ 0 & -b_{12} & -b_{13} & g_4 & g_5 & g_6 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -b_{21} & 0 & 0 \\ g_1 & g_2 & g_3 & 0 & -b_{22} & -b_{23} \end{array}\right]$$

where the values of b_{ij}'s and g_i's are as given above [See (4.3.3)]. The ladder networks associated



NETWORK INTERPRETATION FOR SCHWARZ FORM
FOR MULTIVARIABLE SYSTEMS

FIGURE (4.3.1)

with the subsystems in (4.3.3) are given in Figure (4.3.1). The various quantities L_{11} , C_{12} and L_{13} and L_{21} , C_{22} and L_{23} can be computed respectively from b_{11}, b_{12} and b_{13} and b_{21}, b_{22} and b_{23} using the relations in (4.2.16).

4.4 CONCLUSION

Some possible uses for the network interpretation introduced in this chapter have been given already in section 4.2. Recently Israel Navot [29] has considered the synthesis of certain subclasses of tridiagonal matrices with prescribed eigenvalues. For stable eigenvalues (with negative real parts), Schwarz matrix can be quite simply (and uniquely) synthesized. The network interpretation given in this chapter can then be readily used to "realize" physically the synthesized Schwarz matrix. This results in resistively terminated L-C ladder network. If synthesis in terms of R-L-C ladder networks is contemplated, the necessary methods are given in [27,30,31] . The network interpretation discussed above will be used to justify the procedure employed for simplifying large dynamic systems in chapter V.

CHAPTER - V

A METHOD FOR SIMPLIFYING LARGE DYNAMIC SYSTEMS

5.1 INTRODUCTION

One of the promising features of state-space techniques is that they are applicable to systems (single-variable or multivariable) of any order. However, exact analysis of large dynamic systems (a chemical process control system, for example) demands excessive computational time and cost. Thus it is usual in control systems practice to analyse high order systems through approximate, low order systems. For instance, aircraft systems are often analysed through second order or third order models.

The computational aspect may now be considered. In a typical computational routine, the integration interval is to be of the order of, or less than the smallest time constant of the system. Thus, for a system having some eigenvalues much larger than others (larger eigenvalues mean smaller time constants), such a routine calls for short time increments which would mean an increased cost of computation as the number of steps in the computation will be large with short time increments. In such cases, the modes associated with "large" eigenvalues (high frequency modes) may be neglected, thus reducing the computational complexity. In this work, such an approximation will be called as dominant-mode approximation,

even though it is customary to define a pole (or a pair of poles) to be dominant if it (or they) lies (or lie) nearest the imaginary axis of the s-plane [32,39].

Recently there have been many papers dealing with the simplification of large dynamic systems [33,34,35,36,37, 38]. Davison's method [33] requires the knowledge of the eigenvalues and eigenvectors of the original system in order to be able to determine the system and input matrices for the simplified system. Gustafson [25] employs truncated and associated transfer functions to approximate any order system by a second order system. These methods do not require the computation of the eigenvalues and eigenvectors of the original system. The matrix generalization of the truncated transfer function method through phase-variable canonical form is given in [36]. Here, the Schwarz canonical form is used for the matrix generalization of the associated transfer function method.

All simplification procedures which retain the dominant eigenvalues (with or without modification to account for the neglected non-dominant eigenvalues) suffer from the serious drawback that the approximation made is not good for all control inputs. This can be best illustrated by means of a simple example. Consider a system described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -101 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} u(t)$$

The eigenvalues of this system are -1 and -100. When $u(t)$ is a step function,

$$x_1(t) = 1 - 0.99\, e^{-t} + \frac{1}{99}\, e^{-100t}$$

A dominant-pole approximation of the above system [40] yields

$$\dot{x}^* = x^* + u(t)$$

where $*$ denotes that the variable is only approximate. That is, the mode corresponding to -100 has been neglected. With the same unit step input,

$$x^*(t) = 1 - e^{-t}.$$

Note from the expressions for $x_1(t)$ and $x^*(t)$ that $x_1(t)$ and $x^*(t)$ are reasonably close. Considering the s-plane, the step input $u(t)$ introduces a pole at the origin close to -1 and hence the approximation is reasonable. Suppose $u(t) = e^{-99t}$. This input function introduces a pole at -99 close to -100. Now, with this $u(t)$

$$x_1(t) = \frac{100}{98 \times 99}\, e^{-t} - \frac{100}{98}\, e^{-99t} + \frac{100}{99}\, e^{-100t}$$

and

$$x^*(t) = \frac{1}{98} e^{-t} - \frac{1}{98} e^{-99t}$$

The strength of the mode associated with -100 in the expression for $x_1(t)$ is now much larger than that associated with -1. However, the former mode decays more rapidly than the latter one. Thus, for $t \gg 1/100$ second, $x^*(t)$ is "close" to $x_1(t)$. If the transient response in the interval $[0, 0.001]$ is of interest, then the mode of -100 becomes dominant. The simplification procedure, to be described in this work, also suffers from the above drawback. However, after "sufficiently large" time, the approximate transient response would hopefully approach the exact transient response.

Section 2 of this chapter states clearly the type of systems considered for simplification and outlines the simplification procedure. Justification of the proposed method is given in section 3. Sections 4 and 5 illustrate the method through several examples. Section 5 also discusses some difficulties in reducing systems having complex eigenvalues. Problems for future investigation are posed in section 6.

5.2 SIMPLIFICATION PROCEDURE

The system (with all eigenvalues in the left-half plane) to be considered for simplification is described by the transfer function

$$G(s) = \frac{x(s)}{u(s)} = \frac{1}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n} \tag{5.2.1}$$

A few remarks on this transfer function are now in order. $G(s)$ in (5.2.1) is actually a normalized transfer function. In the usual form (not normalized), it can have in the numerator some gain constant k and the coefficient of s^n in the denominator need not be unity. In such a case, the transfer function can be normalized to the form in (5.2.1) through a simple normalization procedure [See Appendix II, equation (A.2.5)]. Next, $G(s)$ does not have zeros (no numerator dynamics). This constraint can be quite easily relaxed. For details refer to Appendix III. Finally, through a simple choice of state-variables, the system in (5.2.1) can be put in matrix form. Define

$$x_1 = x\ , \quad x_2 = \dot{x}\ , \quad \ldots\ , \quad x_n = x^{(n-1)} \tag{5.2.2}$$

The equations (5.2.1) and (5.2.2) yield

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\ldots \tag{5.2.3}$$

or

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{f}\,u \tag{5.2.4}$$

where $\underline{x}$ and $\underline{f}$ are n-vectors and $\underline{A}$ is an $n \times n$ matrix. Thus the system, instead of being given in the form (5.2.1), could as well have been given directly in the form (5.2.4). In fact, $\underline{A}$ need not be in phase-variable form but can be in any arbitrary form. In such a case, a suitable co-ordinate transformation [15,22] is to be applied to (5.2.4) so that the transformed system is in phase-variable form*. Having pointed out this, hereafter $\underline{A}$ will be assumed to be in phase-variable form.

The simplified system is given by

$$\dot{\underline{x}}^* = \underline{A}^*\,\underline{x}^* + \underline{f}^*\,u \tag{5.2.5}$$

where $\underline{x}^*$ and $\underline{f}^*$ are ℓ-vectors ($\ell < n$) and $\underline{A}^*$ an $\ell \times \ell$ matrix. Further, $\underline{f}^*$ has the first $(\ell - 1)$ entries equal to zero and the last one is different from zero and $\underline{A}^*$ is in phase-variable form.

The system in (5.2.4) is transformed to its Schwarz form, simplification is performed at this stage and the simplified system is then transformed back to the form in (5.2.5). This is the philosophy of the simplification procedure. This will be now elaborately explained.

---

*Such a transformation requires that the system must be completely controllable.

Seek a transformation

$$\underline{z} = \underline{P}\ \underline{x} \tag{5.2.6}$$

which takes (5.2.4) to the form

$$\dot{\underline{z}} = \underline{B}\ \underline{z} + \underline{f}\ u \tag{5.2.7}$$

where $\underline{B}$ is in Schwarz form i.e.

$$\underline{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_2 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -b_{n-1} & -b_n \end{bmatrix} \tag{5.2.8}$$

Compare now b_{n-1} with b_{n-2}. Is b_{n-1} much larger than b_{n-2}? The comparison is said to be successful if $\frac{b_{n-1}}{b_{n-2}} \geqslant 10$. This constant 10 has been found to be suitable for many problems. Whether this comparison is successful or not, go to the next comparison i.e. compare b_{n-2} and b_{n-3}. The i-th comparison is successful if $\frac{b_{n-i}}{b_{n-i-1}} \geqslant 10$. If during the process, $\frac{b_{n-i}}{b_{n-i-1}}$ becomes less than 1, stop comparing any further. If this situation does not happen, comparisons are to be carried on till the step in which b_2 and b_1 are compared. Now the statement of the result is in order. If no comparison turns out successfully, infer that no simplification is possible. If i-th comparison is the last successful one (some or all comparisons before

this i-th successful comparison could have been unsuccessful), then the system in (5.2.7) can be simplified and the simplified system will have the size equal to (n-i). Let $\ell = (n-i)$. The simplified system in Schwarz form is given by

$$\dot{\underline{z}}^* = \underline{B}^* \, \underline{z}^* + \underline{f}^* \, u \tag{5.2.9}$$

where

$$\underline{B}^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -b_1^* & 0 & 1 & \cdots & 0 & 0 \\ 0 & -b_2^* & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -b_{\ell-1}^* & -b_\ell^* \end{bmatrix} \tag{5.2.10}$$

$$\underline{f}^* = (\, 0, \, 0, \, \ldots \, , \, 0, \, p \,)^T \tag{5.2.11}$$

$$\left.\begin{aligned} b_i^* &= b_i \;, \quad i = 1, 2, \ldots, (\ell-1) \,, \\ b_\ell^* &= \frac{b_\ell}{b_{\ell+1}} \; \frac{b_{\ell+2} \cdots}{b_{\ell+3} \cdots} \end{aligned}\right\} \tag{5.2.12}$$

$$p = \frac{1}{b_{\ell+1} \; b_{\ell+3} \cdots} \tag{5.2.13}$$

Transform (5.2.9) to (5.2.5) through

$$\underline{z}^* = \underline{P}^* \; \underline{x}^* \tag{5.2.14}$$

Several comments on this method will now be made. First, $\underline{P}$ and $\underline{P}^*$ can be very easily formed through the procedure given in chapter II. Second, the above method is a matrix generalization of the associated transfer function method discussed in [25]. Reference [25] considers a transfer function G(s) given in (5.2.1) and reduces this to:

$$G'(s) = \frac{1}{R_{(n-1)1}\, s^2 + R_{n1}\, s + R_{(n+1)1}} \tag{5.2.15}$$

where R_{ij}'s are the elements of the Routh array of the denominator polynomial of G(s). To illustrate the idea that the proposed method is a generalization of the associated transfer function approach, let us write down the simplified transfer function (of order ℓ).

$$G^*(s) = \frac{x(s)}{u(s)} \quad \frac{1}{R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}\, s^{\ell} + R_{\cancel{(\ell+2)1}\,(n-\ell+2)1}\, s^{\ell-1} + \cdots + R_{(n+1)1}}$$

or

$$G^*(s) = \frac{1/R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}}{s^{\ell} + \frac{R_{\cancel{(\ell+2)1}\,(n-\ell+2)1}}{R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}}\, s^{\ell-1} + \cdots + \frac{R_{(n+1)1}}{R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}}} \tag{5.2.16}$$

(5.2.16) can be recast as

$$\frac{x(s)}{\left(\dfrac{u(s)}{R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}}\right)} = \frac{1}{s^{\ell} + a_1^{*}\, s^{\ell-1} + \cdots + a_{\ell}^{*}} \qquad (5.2.17)$$

where

$$\left.\begin{aligned} a_1^{*} &= \frac{R_{\cancel{(\ell+2)1}\,(n-\ell+2)1}}{R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}} \\ &\;\vdots \\ a_{\ell}^{*} &= \frac{R_{(n+1)1}}{R_{\cancel{(\ell+1)1}\,(n-\ell+1)1}} \end{aligned}\right\} \qquad (5.2.18)$$

Note here that the factor multiplying $u(s)$ in (5.2.17) $[$i.e. $1/R_{\cancel{(\ell+1)1}}^{(n-\ell+1)1}]$ is equal to p in (5.2.13). The relations (5.2.12) may be easily obtained from (5.2.18).

Thirdly, we shall illustrate as to what such an approximation means. For convenience, consider a fourth order system given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_1 & 0 & 1 & 0 \\ 0 & -b_2 & 0 & 1 \\ 0 & 0 & -b_3 & -b_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (5.2.19)$$

Suppose $(b_3 / b_2) \geqslant 10$. So it is inferred that this system can be approximated by a system which is at least one order less than the system in (5.2.19). Instead of comparing b_2 with b_1 right now, the system in (5.2.19) may be reduced by one order and then b_2 / b_1 may be

computed. As per equations (5.2.9) - (5.2.13), the simplified system is given by

$$\begin{bmatrix} \dot{z}_1^* \\ \dot{z}_2^* \\ \dot{z}_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -b_1 & 0 & 1 \\ 0 & -b_2 & -\frac{b_3}{b_4} \end{bmatrix} \begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{b_4} \end{bmatrix} u \tag{5.2.20}$$

This could have been obtained if we had equated $\dot{z}_4$ in (5.2.19) to zero i.e.

$$\dot{z}_4 = 0 = -b_3\, z_3 - b_4\, z_4 + u$$

$$z_4 = -(b_3 / b_4) z_3 + (1/b_4) u \tag{5.2.21}$$

Substituting (5.2.21) in (5.2.19) results in (5.2.20). Next, if $b_2 / b_1 \geqslant 10$, then a similar procedure can be used to get the 2×2 simplified system. This viewpoint helps us to approximately estimate the variable corresponding to the neglected mode (in this case, z_4). As we have forced $\dot{z}_4$ to 0, the value of z_4 found through (5.2.21) may not represent a good approximation of actual z_4. However, for the sake of completeness, let us investigate the following. Suppose the output equation of the system in (5.2.4) is given as

$$\underline{y} = \underline{C}\ \underline{x} + \underline{d}\ u \tag{5.2.22}$$

Denote $\hat{\underline{y}}$ as the approximate estimate of the output $\underline{y}$, to be evaluated from the simplified system in (5.2.5).

Denote $\hat{\underline{z}}$ as the approximate estimate of $\underline{z}$ in (5.2.7). Then, it may be verified that

$$\hat{\underline{z}} = \underline{K}\,\underline{z}^* + \underline{e}\,u \tag{5.2.23}$$

where

$$\left.\begin{aligned}
\underline{K}_{n\times\ell} &= \underline{K}_{1\,n\times n}\ \underline{K}_{2\,n\times\ell} \\
(\underline{K}_1)_{ij} &= 0\ ,\ i \neq j \\
(\underline{K}_1)_{ii} &= 1\ ,\ i = 1,2,\ldots,\ell \\
(\underline{K}_1)_{(\ell+1)(\ell+1)} &= -\frac{b_\ell\ b_{(\ell+2)}\cdots}{b_{(\ell+1)}\ b_{(\ell+3)}\cdots} \\
(\underline{K}_1)_{(\ell+2)(\ell+2)} &= b_\ell \\
(\underline{K}_1)_{(\ell+i)(\ell+i)} &= b_i\ (\underline{K}_1)_{(\ell+i-2)(\ell+i-2)} \\
&\qquad i = 3,4,\ldots,n-\ell \\
\underline{K}_2 &= \left[\begin{array}{c|c} \underline{I}_{(\ell-1)(\ell-1)} & \underline{0}_{(\ell-1)1} \\ \hline \underline{0}_{(n-\ell+1)(\ell-1)} & \underline{1}_{(n-\ell+1)1} \end{array}\right] \\
\underline{1} &= (\ 1,\ 1,\ \ldots,\ 1,\ 1\ )^T \\
\underline{e}_{n\times 1} &= \underline{E}_{n\times n}\ \underline{e}_{1\,n\times 1} \\
E_{ij} &= 0 \quad i \neq j \\
E_{ii} &= 1\ ,\quad i = 1,2,\ldots,\ell \\
E_{(\ell+1)(\ell+1)} &= \frac{1}{b_{(\ell+1)}\,b_{(\ell+3)}\cdots} \\
E_{(\ell+i)(\ell+i)} &= \begin{cases} 0\ , & i \text{ even} \\ b_i\ E_{(\ell+i-2)(\ell+i-2)}, & i \text{ odd} \end{cases} \\
&\qquad i = 2,3,\ldots,n-\ell \\
\underline{e}_1 &= \begin{bmatrix} \underline{0}_{\ell\times 1} \\ \hline \underline{1}_{(n-\ell)\times 1} \end{bmatrix} . \\
\underline{1}_{(n-\ell)\times 1} &= (\ 1,\ 1,\ \ldots,\ 1,1\ )^T
\end{aligned}\right\} \tag{5.2.24}$$

Again, as mentioned above, $\hat{z}_{\ell+1}$, $\hat{z}_{\ell+2}$,..., $\hat{z}_n$ will be "very" approximate, as we had equated the derivatives of these variables to zero. Now, as

$$\underline{x} = \underline{P}^{-1} \underline{z} \quad , \tag{5.2.25}$$

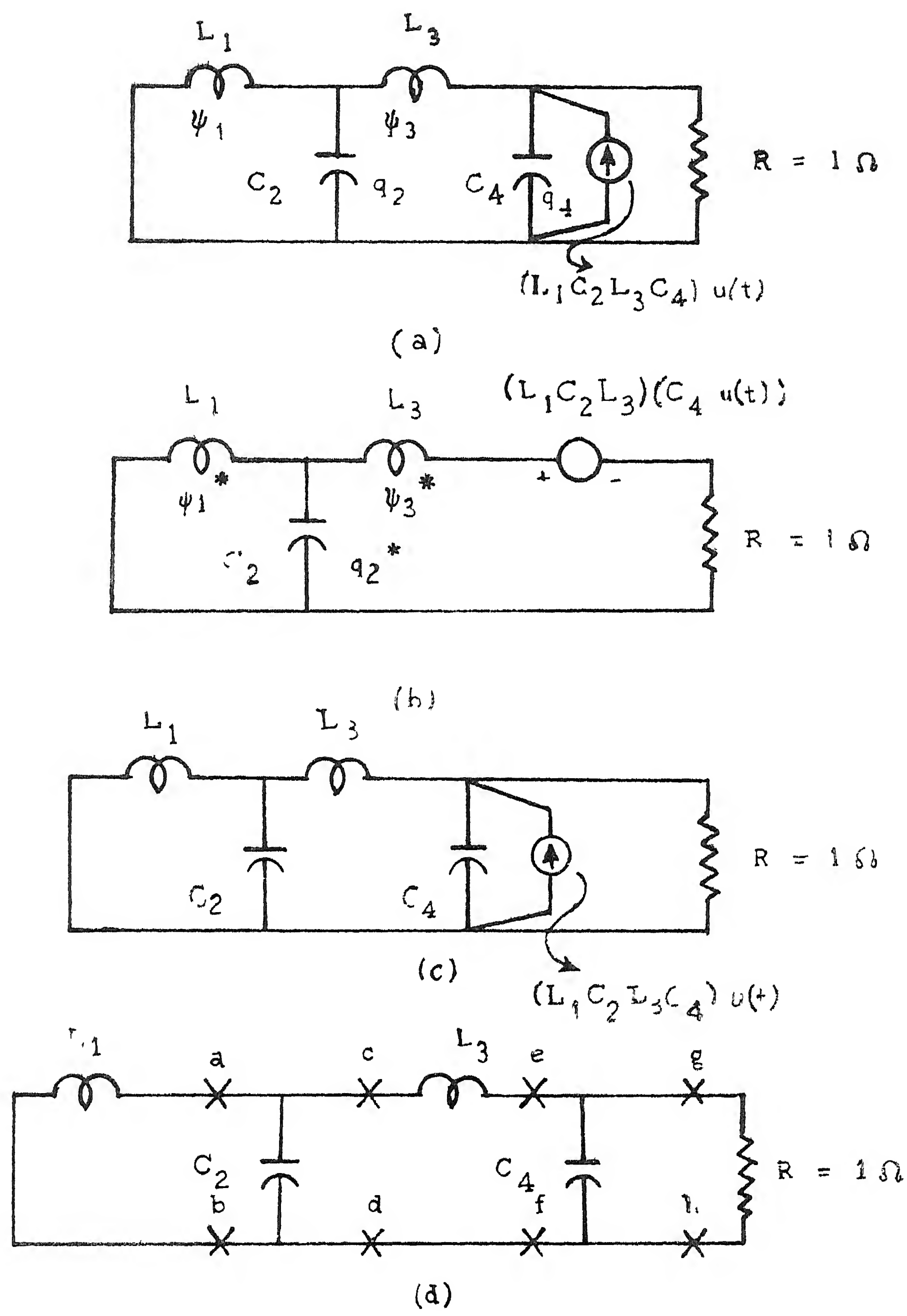
$$\hat{\underline{x}} = \underline{P}^{-1} \hat{\underline{z}} = \underline{P}^{-1} \underline{K} \underline{z}^* + \underline{P}^{-1} \underline{e} u \tag{5.2.26}$$

(Note carefully that $\hat{\underline{x}}$ and $\hat{\underline{z}}$ are n-vectors and $\underline{x}^*$ and $\underline{z}^*$ are ℓ-vectors). Using (5.2.22) and (5.2.14)

$$\left.\begin{aligned} \hat{\underline{y}} &= \underline{C}^* \underline{x}^* + \underline{d}^* u \\ \text{where} \\ \underline{C}^* &= \underline{C} \underline{P}^{-1} \underline{K} \underline{P}^* \\ \text{and} \\ \underline{d}^* &= \underline{C} \underline{P}^{-1} \underline{e} + \underline{d} \end{aligned}\right\} \tag{5.2.27}$$

It is worthwhile to repeat that $\hat{\underline{y}}$ will be "very" approximate and will represent a reasonable approximation of $\underline{y}$ [actual value determined by (5.2.22)] if $\underline{C}$ in (5.2.22) is such that $\underline{y}$ depends only on the first ℓ -components of the state vector $\underline{x}$.

Fourth, we shall make use of network interpretation concept discussed in chapter IV to investigate this approximation process. Again, for convenience of explanation, consider a fourth order system given in (5.2.19). This system can be modeled by the ladder network shown in Figure (5.2.1-a). The state-variables associated with the network (marked in the Figure) and



SYSTEM SIMPLIFICATION EXPLAINED THROUGH NETWORK INTERPRETATION

FIGURE (5.2.1)

the z_i's $(i=1,2,3,4)$ are related by:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{L_1} & & & \\ & \frac{1}{L_1C_2} & \underline{0} & \\ & & -\frac{1}{L_1C_2L_3} & \\ & \underline{0} & & \frac{1}{L_1C_2L_3C_4} \end{bmatrix} \begin{bmatrix} \psi_1 \\ q_2 \\ \psi_3 \\ q_4 \end{bmatrix} \quad (5.2.28)$$

The state equations for the ladder network can be written as:

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{q}_2 \\ \dot{\psi}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1/C_2 & 0 & 0 \\ 1/L_1 & 0 & -1/L_3 & 0 \\ 0 & 1/C_2 & 0 & -1/C_4 \\ 0 & 0 & 1/L_3 & -1/C_4 \end{bmatrix} \begin{bmatrix} \psi_1 \\ q_2 \\ \psi_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ L_1C_2L_3C_4 \end{bmatrix} u$$

... (5.2.29)

Suppose $(b_3/b_2) \geqslant 10$. As $b_3 = 1/(L_3C_4)$ and $b_2 = 1/(C_2L_3)$, this inequality implies $C_4/C_2 \leq 10$. Now, equate $\dot{q}_4$ to zero, thus find q_4 in terms of ψ_3 and u and substitute back in (5.2.19) to get

$$\begin{bmatrix} \dot{\psi}_1^* \\ \dot{q}_2^* \\ \dot{\psi}_3^* \end{bmatrix} = \begin{bmatrix} 0 & -1/C_2 & 0 \\ 1/L_1 & 0 & -1/L_3 \\ 0 & 1/C_2 & -1/L_3 \end{bmatrix} \begin{bmatrix} \psi_1^* \\ q_2^* \\ \psi_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -C_4L_1C_2L_3 \end{bmatrix} u$$

... (5.2.30)

Equations (5.2.30) correspond to a ladder network shown in Figure (5.2.1-b). The Schwarz form for the network in Figure (5.2.1-b) can be readily written as

$$\begin{bmatrix} \dot{z}_1^* \\ \dot{z}_2^* \\ \dot{z}_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -b_1 & 0 & 1 \\ 0 & -b_2 & -b_3/b_4 \end{bmatrix} \begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/b_4 \end{bmatrix} u$$

and this is exactly the same as the one obtained through (5.2.9) - (5.2.13) with $\ell=3$ and $n=4$. Remember that $\dot{q}_4$, above, was equated to zero. Thus a rough estimate of q_4 can be found from this equation. It may be verified that this would, after applying the transformation $z_4 = \frac{q_4}{L_1C_2L_3C_4}$ [See equation (5.2.28)] , result in (5.2.21).

A careful observation of Figures (5.2.1-a) and (5.2.1-b) reveals that the network in Figure (5.2.1-b) may be obtained by deleting the component C_4 from the network in Figure (5.2.1-a) and then transforming the current source in Figure (5.2.1-a) to a voltage source as in Figure (5.2.1-b) . This viewpoint [that is, neglecting the $(n-\ell)$ dynamic components near the resistor end of the ladder network when the simplified system size is determined to be ℓ] will be heuristically justified in the next section.

Fifth, it is to be emphasized that the proposed method <u>does not</u> require the computation of eigenvalues

major advantage over Davison's method [33]. However, the method proposed here shows, at present, some difficulties with systems having both real and complex eigenvalues. These are elaborately discussed in section 6. It is hoped that in future, these difficulties would be resolved.

Finally, a couple of more favourable comments will be made. Construction of Liapunov function for the reduced or simplified system is quite trivial. Denoting V^* as the required Liapunov function,

$$V^* = \underline{x}^{*T} (\underline{P}^{*T} \underline{L}^* \underline{P}^*) \underline{x}^* \qquad (5.2.31)$$

where

$$\underline{L}^* = \text{diag.}\left\{(b_1^* b_2^* \ldots b_\ell^*), (b_2^* b_3^* \ldots b_\ell^*), \ldots, (b_{\ell-1}^* b_\ell^*), b_\ell^*\right\} \qquad (5.2.32)$$

Davison's method results in a reduced system matrix $\underline{A}^*$ which retains all the dominant eigenvalues of $\underline{A}$ <u>without</u> modification. But the method proposed here modifies the eigenvalues in order to account for the neglected modes. It is believed that this method would yield improved approximation. Also note that if the original system is asymptotically stable*, the simplified system is also asymptotically stable as all the b_i^*'s $(i=1,2,\ldots,\ell)$ are > 0.

---

* In this work, only asymptotically stable systems are of interest. This fact has been mentioned in the beginning of this section.

5.3 JUSTIFICATION OF THE SIMPLIFICATION PROCEDURE

This section aims to show heuristically that whenever the given (original) system has i eigenvalues which are "much larger"* than the smallest eigenvalue of the system, the i-th comparison will be the last successful one (refer to section 5.2). As the b_i's $(i=1,2,\ldots,n)$ of the Schwarz matrix depend nonlinearly (through Hurwitz determinants) upon the coefficients of the characteristic polynomial and these latter coefficients themselves are related to eigenvalues in a complicated manner (Any book on theory of equations will give these relations), a rigorous justification of the above statement may not be possible. Instead, a heuristic reasoning will be provided in what follows. Generalization, based on the experience gained by considering low order systems, of the results to any n-th order system will be made. Before proceeding with the heuristic justification, it is to be mentioned once again that the whole simplification procedure works very well with real eigenvalue systems (repeated or distinct), but encounters some difficulties with systems having real and complex eigenvalues.

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*An eigenvalue is said to be large if its magnitude is more than say k times that of the smallest eigenvalue of the system, where the value of k has to be decided depending upon the accuracy desired. A convenient figure

Consider a 3x3 system. It is assumed that the eigenvalues λ_1, λ_2 and λ_3 of this system are real and stable, and are such that one of them (magnitude only), say λ_1, is much larger than the others i.e. λ_2 and λ_3. This implies that if

$$\left.\begin{aligned} \lambda_2 &= k_1 \lambda_1 \\ \lambda_3 &= k_2 \lambda_1 \end{aligned}\right\} \qquad (5.3.1)$$

then, k_1 and $k_2 << 1$. Let $r_i = -\lambda_i$ $(i=1,2,3)$*. If the characteristic polynomial of this system is given as

$$P(s) = s^3 + a_1 s^2 + a_2 s + a_3 , \qquad (5.3.2)$$

then

$$b_1 = a_3/a_1 \; ; \quad b_2 = a_2 - a_3/a_1$$

$$\text{and} \quad b_3 = a_1 \qquad (5.3.3)$$

From the relations between eigenvalues and the coefficients of $P(s)$, (5.3.3) can be rewritten as:

$$\left.\begin{aligned} b_1 &\cong k_1 k_2 r_1^2 \\ b_2 &\cong (k_1 + k_2) r_1^2 \\ b_3 &= (1 + k_1 + k_2) r_1 \end{aligned}\right\} \qquad (5.3.4)$$

---

*The relationships between the λ_i's and the coefficients of the characteristic polynomial involve powers of (-1). So to avoid such factors, r_i's are defined and now the resulting relations will be simpler [41] .

As k_1 and $k_2 \ll 1$, it is obvious from (5.3.4) that

$$\frac{b_2}{b_1} \gg 1 \qquad (5.3.5)$$

thus resulting in one successful comparison. When $k_1 = k_2 \ll 1$ (i.e. λ_2 and λ_3 are equal and they are much less than λ_1), (5.3.5) still holds. Suppose k_1 is comparable to 1. Then for $k_2 \ll 1$

$$\frac{b_2}{b_1} = \frac{k_1 + k_2}{k_1 k_2} = \frac{1}{k_1} + \frac{1}{k_2} > \frac{1}{k_2} \gg 1$$

Therefore, even in this case (5.3.5) holds.

Let us now consider a fourth order system. With the same notations as used in the previous case, the coefficients of the characteristic polynomial can be expressed in terms of the r_i's $(i=1,2,3,4)$ as under:

$$\left.\begin{aligned}
a_1 &= [1 + (k_1 + k_2 + k_3)]\, r_1 \\
a_2 &= [(k_1 + k_2 + k_3) + (k_1k_2 + k_1k_3 + k_2k_3)]\, r_1^2 \\
a_3 &= [(k_1k_2 + k_1k_3 + k_2k_3) + k_1k_2k_3]\, r_1^3 \\
a_4 &= (k_1k_2k_3)\, r_1^4
\end{aligned}\right\} \qquad (5.3.6)$$

Also,

$$\left.\begin{aligned}
b_1 &= a_4/(a_2 - a_3/a_1) \\
b_2 &= a_3/a_1 - a_4/(a_2 - a_3/a_1) \\
b_3 &= a_2 - a_3/a_1 \\
b_4 &= a_1
\end{aligned}\right\} \qquad (5.3.7)$$

Now

$$b_3 = a_2 - a_3/a_1 = r_1^2 \Big[(k_1+k_2+k_3)+(k_1k_2+k_1k_3+k_2k_3) - (k_1k_2+k_1k_3+k_2k_3+k_1k_2k_3)/(1+k_1+k_2+k_3) \Big] \quad \ldots(5.3.8)$$

$$b_2 = a_3/a_1 - a_4/(a_2 - a_3/a_4)$$

$$= r_1^2 \left[(k_1k_2 + k_1k_3 + k_2k_3 + k_1k_2k_3)/(1+k_1+k_2+k_3) - \frac{k_1k_2k_3(1+k_1+k_2+k_3)}{(k_1+k_2+k_3)(1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3-k_1k_2k_3)} \right] \quad \ldots(5.3.9)$$

Having written down the exact expressions for b_2 and b_3, let us investigate the following cases.

Case I: Suppose k_1, k_2 and $k_3 \ll 1$. That is, λ_2, λ_3 and λ_4 are much smaller than λ_1. Then from (5.3.8),

$$b_3 \cong r_1^2 (k_1 + k_2 + k_3) \quad (5.3.10)$$

In arriving at (5.3.10) from (5.3.8), two approximations have been made. They are:

(i) $k_1k_2k_3$ is neglected in comparison with $(k_1 + k_2 + k_3)$ and $(k_1k_2 + k_1k_3 + k_2k_3)$.

(ii) $(k_1k_2 + k_1k_3 + k_2k_3)$ is neglected in comparison with $(k_1 + k_2 + k_3)$.

These approximations are reasonable as k_1, k_2 and k_3 are $\ll 1$. Neglecting similarly $k_1 k_2 k_3$ and

$(k_1k_2 + k_1k_3 + k_2k_3)$ in comparison with $(1+k_1+k_2+k_3)$ in (5.3.9) we get

$$b_2 \doteq r_1^2 \left[\frac{k_1k_2+k_1k_3+k_2k_3}{1+k_1+k_2+k_3} - \frac{k_1k_2k_3(1+k_1+k_2+k_3)}{(k_1+k_2+k_3)(1+k_1+k_2+k_3)} \right] \quad (5.3.11)$$

(5.3.11) can be rewritten as

$$b_2 = \left[\frac{r_1^2}{1+k_1+k_2+k_3} \right] \left\{ k_1k_2 + k_1k_3 + k_2k_3 - \frac{k_1k_2k_3}{(k_1+k_2+k_3)} \right\} \quad (5.3.12)$$

Thus,

$$b_2 < \frac{r_1^2}{1+k_1+k_2+k_3} \quad (k_1k_2 + k_1k_3 + k_2k_3) \quad (5.3.13)$$

From (5.3.10) and (5.3.13), it is easily seen that

$$\frac{b_3}{b_2} > \frac{(k_1+k_2+k_3)(1+k_1+k_2+k_3)}{(k_1k_2+k_1k_3+k_2k_3)} > \frac{(k_1+k_2+k_3)}{(k_1k_2+k_1k_3+k_2k_3)} \quad (5.3.14)$$

As k_1, k_2 and k_3 are $\ll 1$, (5.3.14) implies

$$\frac{b_3}{b_2} \gg 1 \quad (5.3.15)$$

Recapitulating, we had a fourth order system with one eigenvalue much larger than others and hence we obtained a successful comparison given in (5.3.15).

<u>Case II</u>: Suppose k_1, k_2 and k_3 are all not much less than one i.e. for instance $k_1 \cong 1$, but k_2 and k_3 may be much less than one. Even in this, going through the steps given in Case I, it is easily seen that b_3 is larger than b_2 $(b_3 > b_2)$ but not necessarily much larger than b_2. Also, it is readily seen from (5.3.7) that

$$1 + b_2 / b_4 = a_3(a_2 - a_3/a_1)/a_1a_4 \qquad (5.3.16)$$

Substituting for a_i's in (5.3.16) and rearranging yield

$$1 + \frac{b_2}{b_1} = \left\{\left(1 + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)\left\{\frac{(k_1 + k_2 + k_3)}{(1+k_1+k_2+k_3)^2}\right. \right.$$
$$\left.\left.\left[1+k_1+k_2+k_3+k_1k_2+k_1k_3+k_2k_3-k_1k_2k_3(1+k_1+k_2+k_3)\right]\right\}\right\} . \qquad \ldots \quad (5.3.17)$$

Now making use of the fact that $k_1 \cong 1$, k_2 and $k_3 \ll 1$ and after some simple manipulations we obtain

$$1 + b_2/b_1 = (k_2 + k_3) / 2k_2k_3 \qquad (5.3.18)$$

As k_2 and $k_3 \ll 1$, (5.3.18) implies

$$b_2/b_1 \gg 1 \qquad (5.3.19)$$

(For example, if $k_2 = 0.01$, $k_3 = 0.02$ and $k_1 \cong 1$, b_2/b_1 becomes approximately 75). Summarizing, we had a fourth order system with two eigenvalues comparable to each other in magnitude and much larger than the other two and this resulted in a situation wherein the second

comparison was the last successful one.

Case III: Suppose the system has two distinct real eigenvalues and one repeated eigenvalue of multiplicity 2. Two possibilities are now open to us. In one, the repeated eigenvalues have to be retained in the reduced system. In the second case, the repeated eigenvalues have to be neglected. From the results of Case I and Case II, it may be easily seen that in the first possibility, two successful comparisons will be obtained and in the latter, two successful comparisons need not occur but the second comparison ($b_2/b_1 \gg 1$) will be successful. Thus in either case the proposed method will reduce the system by two order.

In a general n-th order case also, the same procedure can be applied to verify the validity of the proposed method (of course in real eigenvalue case only). As expected, the expressions for b_i's now become quite involved, and thus are not given in this section.

In section 5.2, it is asserted that whenever we have the i-th comparison as the last successful one, the i dynamic components near the resistor end of the ladder network can be neglected. Let us heuristically justify this assertion. In view of the discussion presented above, we have thus i eigenvalues which are "large" compared to others.

Gustafson [44] states that the "high frequency modes" accumulate near the resistor end of the ladder network (See section 5.2 for the Figures) and thus neglecting the dynamic components near the resistor end only eliminates the high frequency modes. (Note that the method proposed in this work gives a way to determine how many such dynamic components near the resistor end are to be neglected). Let us in what follows attempt to "prove"* Gustafson's statement at least in real eigenvalue case+.

For convenience of explanation, consider a fourth order system. The corresponding ladder network for this system is shown in Figure (5.2.1-c). Now consider the Figure (5.2.1-d). Looking into the port 'ab' from right, we have L_1 and this resonates at the frequency $f_1 = 0$. Look now into the port 'cd'. We have L_1 and C_2 in parallel. This parallel combination resonates at a frequency, say, f_2 and $f_2 > 0$. Let us now look into the port 'ef'. L_1 and C_2 are in parallel and this

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*This heuristic proof was suggested to the author by Professor S. Venkateswaran, Department of Electrical Engineering, Indian Institute of Technology, Kanpur.

+It is to be observed that Gustafson's statement is for any general system whether it has real or complex eigenvalues.

combination is in series with L_3. At frequencies higher than f_2, the parallel combination of L_1 and C_2 becomes capacitive and this capacitance (call this as residual capacitance C_{2r}) is less than C_2 (as L_1 neutralizes a part of C_2. In fact, at f_2, the effective admittance ~~impedance~~ of L_1 in parallel with C_2 is equal to zero). It may be observed that for systems with real eigenvalues $L_i > L_{i+2}$, $(i=1,3,\ldots)$ and $C_i > C_{i+2}$, $(i=2,4,\ldots)$. Thus $L_3 < L_1$ and $C_{2r} < C_2$. Hence the resonating frequency of the circuit located to the left of 'ef', say f_3, is larger than f_2 i.e. $f_3 > f_2$. Similarly we can show that the resonating frequency f_4 of the circuit located to the left of 'gh' is larger than f_3. Hence we have

$$f_4 > f_3 > f_2 > f_1 \tag{5.3.20}$$

These frequencies roughly correspond to the eigenvalues of the system. Thus we infer that the dynamic elements responsible for the high frequency modes are located near the resistor end. The arguments presented above can be very easily extended to any n-th order system with real eigenvalues.

Is Gustafson's statement true for systems with both complex and real eigenvalues ? For such systems, the statement that $L_i > L_{i+2}$ and $C_i > C_{i+2}$ is not necessarily true. Let us give an example to illustrate this.

Consider

$$G(s) = \frac{1}{s^4 + 0.1s^3 + 100.01s^2 + 10.1s + 1}$$

For this system, $L_1 = 0.1H$, $L_3 = 10H$, $C_2 = 0.1F$ and $C_4 = 10F$. Thus in this system $L_1 \not> L_3$ and $C_2 \not> C_4$.

Suppose in the case of the fourth order system $b_3 \gg b_2$ i.e. $C_4 \ll C_2$. It is said above that C_4 can be neglected. Are we justified in doing so ? The answer is yes if we can show under the condition $C_4 \ll C_2$ that the impedance looking into 'ef' is much less than the impedance due to C_4 [See Figure (5.2.1-d)] . That is, we must have $|Z_1(s)|$ much less than $\left|\frac{1}{C_4 s}\right|$ where

$$Z_1(s) = L_3 s + \frac{1}{C_2 s + \frac{1}{L_1 s}} = L_3 s + \frac{L_1 s}{1 + L_1 C_2 s^2}$$

or

$$\left| L_3 C_4 s^2 + \frac{L_1 C_4 s^2}{1 + L_1 C_2 s^2} \right| \ll 1$$

As $L_1 C_4 \ll L_1 C_2$, this becomes

$$\left| L_3 C_4 s^2 \right| \ll 1$$

Or

$$L_3 C_4 \omega^2 \ll 1 \qquad (5.3.21)$$

Now

$$\omega^2_{min} = \frac{1}{L_{max}\,C_{max}} = \frac{1}{(L_1+L_3)(C_2+C_4)}$$

$$\doteq \frac{1}{C_2(L_1+L_3)} \quad \text{as} \quad C_2 \gg C_4$$

When $\omega = \omega_{min}$ is substituted in $L_3C_4\omega^2$, we obtain

$$L_3C_4\omega^2_{min} = \frac{L_3\,C_4}{(L_1 + L_3)C_2} \lll 1$$

as $C_4 \ll C_2$ and $L_3 < L_1+L_3$. Thus at $\omega = \omega_{min}$ and for $C_2 \gg C_4$, $\left|Z_1(s)\right| \ll \left|\frac{1}{C_4 s}\right|$. Of course for all ω, this inequality may not be true. Actually ω_{min} corresponds to maximum time constant. So the approximated system (with C_4 thrown off) response approaches the actual system response as the time becomes larger than τ_{max}, the matimum time constant of the system. Note also that $L_1C_4\omega^2 \ll 1$ may be true for some $\omega > \omega_{min}$, but not for $\omega \gg \omega_{min}$. So, even for $t < \tau_{max}$, the approximated system response is very close to the exact response. When t is very small (just near 0), the high frequency modes do contribute to the transient response. Thus the approximated system response differs much (comparatively) from the exact response, but as time increases this difference in the responses decreases. Thus we can, as an approximation, neglect C_4 when $C_4 \ll C_2$.

Of course the current source in Figure (5.2.1-c) can be transformed into a voltage source to get the network for the approximate system as in Figure (5.2.1-b).

Throughout the discussion in this section, we have tacitly avoided the consideration of systems with real and complex eigenvalues. However, some discussions on such systems will be given in section 5.5.

5.4 EXAMPLE SYSTEMS WITH REAL EIGENVALUES

Sections 4 and 5 aim at illustrating the use of the proposed method. Several example systems are considered and for typical cases, responses of the actual and approximate systems are plotted in order to gain a feel for the extent of approximation made.

Before proceeding with example systems, let us, for convenience, summarize the results stated in section 5.2. Given a stable system described by

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{f}\,u \qquad (5.4.1)$$

with $(\underline{A}\ ,\ \underline{f})$ as in (5.2.3). Compute first the elements (denoted by b_i's , $i=1,2,\ldots,n$) of the Schwarz matrix (denoted by $\underline{B}$) corresponding to $\underline{A}$. Next, carry out the comparison process starting with the comparison of b_{n-1} with b_{n-2}. In general, i-th comparison would involve comparing b_{n-i} with b_{n-i-1} and this i-th comparison is declared successful if the quotient

b_{n-i}/b_{n-i-1} is greater than or equal to k where the value of k to be chosen depends upon the extent of approximation desired* and hence its choice is quite subjective. Unless stated otherwise, $k = 9.5$ is used throughout sections 5.5 and 5.6. A higher value of k would imply a higher order for the approximate system and hence may result in a better approximation. Whenever during the comparison process b_{n-i}/b_{n-i-1} becomes less than one, terminate the comparison process. Otherwise, continue doing the comparisons until b_2 is compared with b_1. If i-th comparison is the last successful comparison, then infer that the actual system can be approximated by a system of order (n-i). The reduced system is described by

$$\dot{\underline{z}}^* = \underline{B}^* \underline{z}^* + \underline{f}^* u \qquad (5.4.2)$$

where $\underline{B}^*$ and $\underline{f}^*$ are as given in (5.2.10) through (5.2.13). Transform the system in (5.4.2) to one in phase-variable canonical form given by

$$\dot{\underline{x}}^* = \underline{A}^* \underline{x}^* + \underline{\hat{f}}^* u \qquad (5.4.3)$$

through a transformation (refer, for details, to chapter III).

$$\underline{z}^* = \underline{P}^* \underline{x}^* \qquad (5.4.4)$$

The system in (5.4.3) is the required, simplified system.

*In a typical dominant-pole approximation method, an eigenvalue which is say 25 times larger than the smallest eigenvalue (excluding zero eigenvalues) may be neglected.

It is worth mentioning once again that our method does not require the computation of eigenvalues and eigenvectors of the actual system. However, to have a feel for what is happening in the approximation process, in all the example systems, we give first the eigenvalues of the original system and then those of the simplified system (Note that our approximation or simplification method, after eliminating the system "far-off eigenvalues", modifies the remaining ones to hopefully get a simplified system with responses "close" to those of the actual system). Comments on the modification (due to our simplification method) of the eigenvalues of the system are also given for many example systems. Finally for typical cases, responses of both actual and simplified systems are plotted.

In this section, we consider a few systems with real eigenvalues. The simplification process for the first example system is explained elaborately and for others, only end results are stated. Also for the first example system, the reduction process is performed through Davison's method [33] and the results of our method and Davison's method are compared. Towards the end of this section, an example system with numerator dynamics is discussed. The simplification of this system is carried out through three different methods and the results are compared.

Example (5.4.1): Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_6 & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad \ldots(5.4.5)$$

where

$a_1 = 36.09999968$

$a_2 = 388.87999488$

$a_3 = 1398.33999360$

$a_4 = 1234.39977472$

$a_5 = 394.00537088$

$a_6 = 40.08072896$

The eigenvalues of this system are:

$\lambda_1 = -0.2$, $\lambda_2 = -0.4$, $\lambda_3 = -0.5$, $\lambda_4 = -5.0$, $\lambda_5 = -10.0$ & $\lambda_6 = -20.0$.

The elements b_i's $(i=1,2,\ldots,6)$ of the corresponding Schwarz matrix are found out as:

$b_1 = 0.03590879$

$b_2 = 0.27054737$

$b_3 = 3.18777084$

$b_4 = 35.24095232$

$b_5 = 350.14481408$

$b_6 = 36.09999968$

Now start the comparison process.

$$b_5/b_4 \cong 10$$
$$b_4/b_3 \cong 11$$
$$b_3/b_2 \cong 11.8$$
$$b_2/b_1 \cong 7.5$$

Note that the first three comparisons are successful and that the fourth one is unsuccessful. Thus, our method indicates that the order of the simplified system is 3. Using equations (5.2.10) through (5.2.13), the simplified system in Schwarz form is written as:

$$\begin{bmatrix} \dot{z}_1^* \\ \dot{z}_2^* \\ \dot{z}_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -b_1^* & 0 & 1 \\ 0 & -b_2^* & -b_3^* \end{bmatrix} \begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix} u(t) \qquad (5.4.6)$$

where

$$b_1^* = 0.0359087$$
$$b_2^* = 0.27054737$$
$$b_3^* = 0.87736430$$

and $p = 0.00078604.$

Apply a transformation

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b_1^* & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1^* \\ z_2^* \\ z_3^* \end{bmatrix} \qquad (5.4.7)$$

to the system in (5.4.6) to result in a system

The responses for unit step input, $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_1^*(t)$, $x_2^*(t)$ and $x_3^*(t)$ are plotted in Figure (5.4.1). Observe that the approximate responses follow closely the actual responses. However, the error associated with derivative variables $x_2^*(t)$ and $x_3^*(t)$ are considerably large compared to that associated with $x_1^*(t)$. This is true for all example systems to be discussed.

Davison's method [33] will now be applied to reduce the sixth order system in (5.4.5) to a third order system. A computer program has been used to compute the reduced system matrices $\underline{A}^{**}$ and $\underline{f}^{**}$. Here, only the final results are given:

$$\underline{A}^{**} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.04 & -0.38 & -1.1 \end{bmatrix}$$

$$\underline{f}^{**} = \begin{bmatrix} 0.0001127 \\ -0.0004648 \\ 0.0014694 \end{bmatrix}$$

and the reduced system is described by:

$$\dot{\underline{x}}^{**} = \underline{A}^{**} \underline{x}^{**} + \underline{f}^{**} u \qquad (5.4.9)$$

Note that $\underline{A}^{**}$ and $\underline{f}^{**}$ have been calculated from the knowledge of eigenvalues and eigenvectors (modal matrix) of $\underline{A}$. (For details refer to [33]). $\underline{A}^{**}$ has eigenvalues -0.2, -0.4 and -0.5.

The approximate system responses $x_1^{**}(t)$, $x_2^{**}(t)$ and $x_3^{**}(t)$ are plotted in the same Figure (5.4.1). Compare the responses of the systems obtained by our method and by Davison's method. $x_1^{**}(t)$ is closer to the actual response $x_1(t)$ than $x_1^{*}(t)$. But $x_2^{*}(t)$ and $x_3^{*}(t)$ are much better approximations than $x_2^{**}(t)$ and $x_3^{**}(t)$. It is to be noted that in Davison's method also, error with the derivative variables $x_2^{**}(t)$ and $x_3^{**}(t)$ is considerably large compared to the error with $x_1^{**}(t)$.

Example (5.4.2): From now on, only final results will be given. Unstarred quantities refer to actual system and the starred ones to approximate system. With this notation, all the results will now be presented.

$$\lambda_1 = -1, \ \lambda_2 = -2, \ \lambda_3 = -50 \ \text{ and } \ \lambda_4 = -50.$$

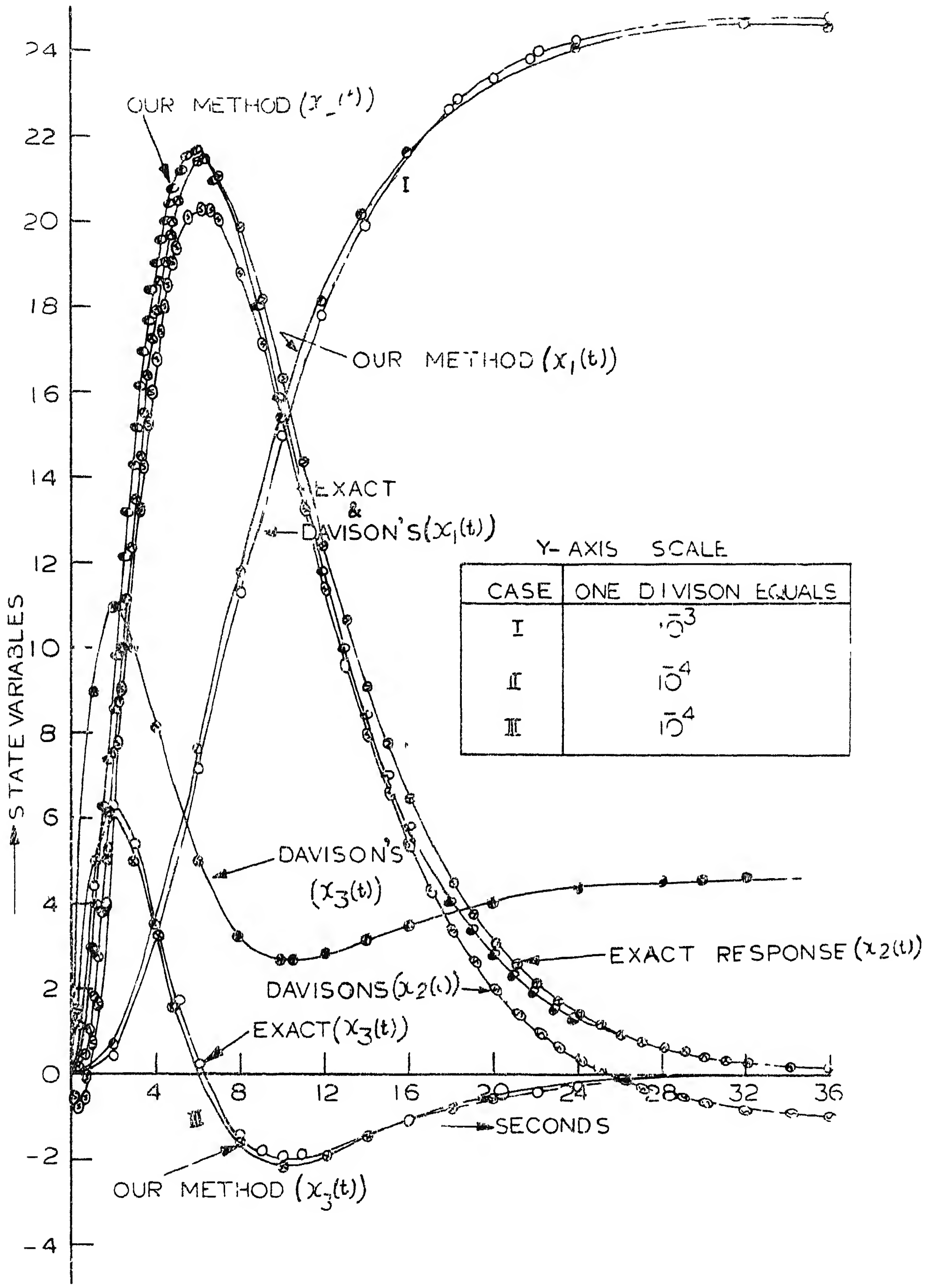
That is, the actual system has a repeated eivenvalue at -50 with multiplicity 2. The comparison process resulted in

$$b_3/b_2 = 38 \quad \text{and} \quad b_2/b_1 = 40$$

thus both the comparisons being successful. The reduced system has eigenvalues:

$$\lambda_1^* = -1.12617126$$
$$\lambda_2^* = -1.62795264$$

The responses of actual and approximate systems, for a unit



EXACT AND APPROXIMATE RESPONSES FOR EXAMPLE (5 4 1)

FIGURE (5 4 1)

step input, are plotted in Figure (5.4.2).

Example (5.4.3): The actual system has

$$\lambda_1 = -50, \; \lambda_2 = -50, \; \lambda_3 = -50 \text{ and } \lambda_4 = -100.$$

We get

$$b_3/b_2 = 6.7 \text{ and } b_2/b_1 = 4.3.$$

So neither of the comparisons is successful and hence no simplification is possible. This conclusion arrived at through our method is consistent with the expected result.

Example (5.4.4): Let us consider a system with eigenvalues "fairly spread".

$$\lambda_1 = -100, \; \lambda_2 = -50, \; \lambda_3 = -30, \; \lambda_4 = -2, \; \lambda_5 = -1 \text{ and } \lambda_6 = -0.1.$$

We get in this case

$$b_5/b_4 = 9 \;, \; b_4/b_3 = 33 \;, \; b_3/b_2 = 10 \text{ and } b_2/b_1 = 20.$$

Even though the first comparison is not successful, all the other three are successful and hence the simplified system is of order two. The eigenvalues of the simplified system are approximately -0.084 and -1.6.

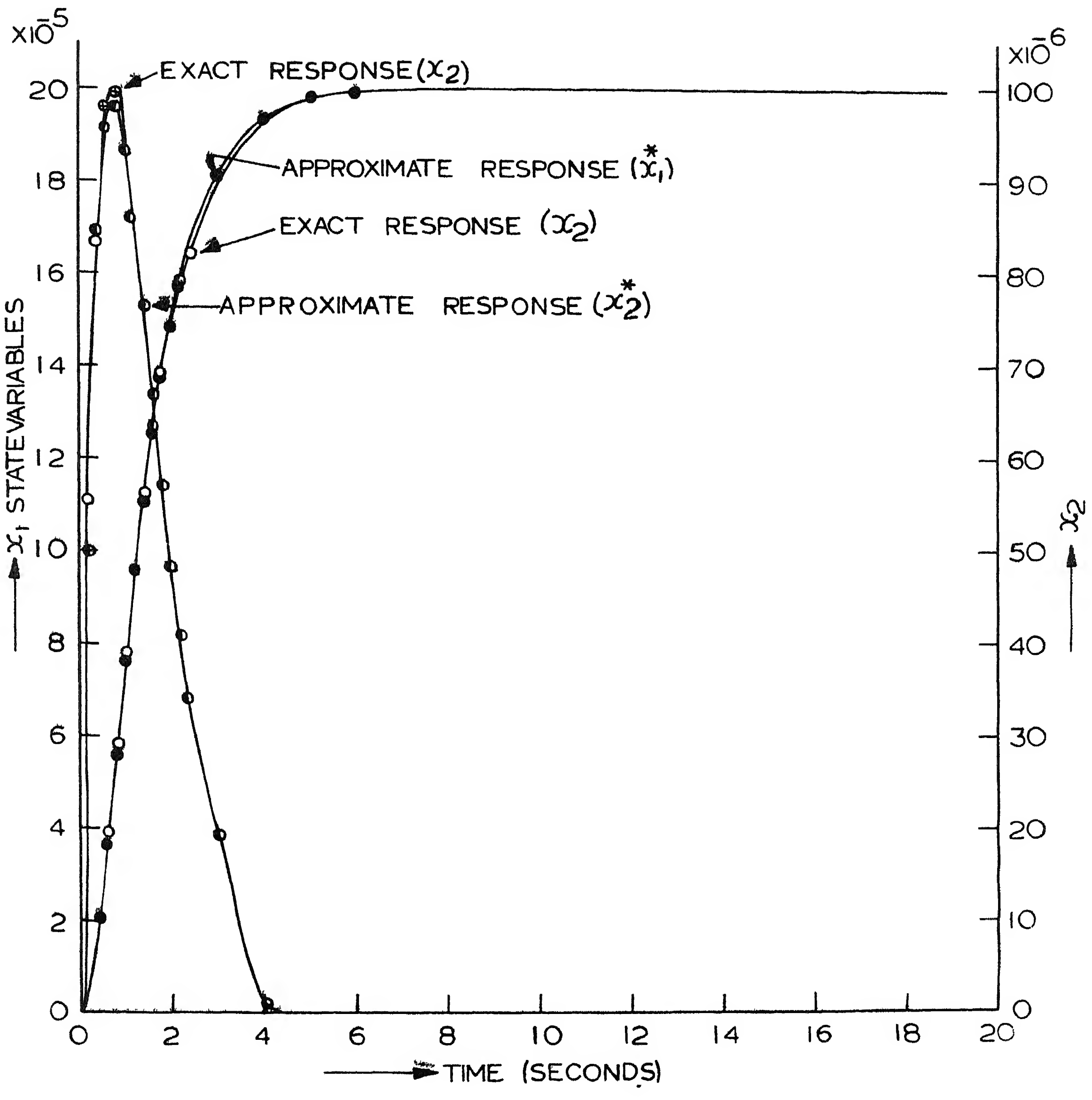
Example (5.4.5): Let us now consider an example with one repeated eigenvalue which is dominant (i.e. to be retained in the simplified system).

$$\lambda_1 = -0.1 \;, \; \lambda_2 = -0.1 \;, \; \lambda_3 = -5 \text{ and } \lambda_4 = -10.$$

A rough calculation indicates that

$$b_3/b_2 = 75 \text{ and } b_2/b_1 = 65$$

i.e. two comparisons are successful. So the simplified



EXACT AND APPROXIMATE RESPONSE FOR EXAMPLE (5 4 2)

FIGURE (5 4 2)

system is of order two.

Let us now consider the last example in this section which involves numerator dynamics.

Example (5.4.6)* Finally, consider a system with numerator dynamics described by

$$\frac{X(s)}{U(s)} = \frac{(s+4)}{(s+1)\ (s+3)\ (s+5)\ (s+10)} \qquad (5.4.10)$$

Following the method II given in Appendix II, the system in (5.4.10) can be equivalently represented by the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -150 & -245 & -113 & -19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad \ldots (5.4.11)$$

$$x = 4x_1 + x_2 = \begin{bmatrix} 4 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \ldots (5.4.12)$$

where $x(t) = L^{-1}[(X(s))]$.

The elements b_i's ($i=1,2,3,4$) for (5.4.11) are computed and the comparison process results in

$b_3/b_2 = 8.8$ and $b_2/b_1 = 7.6$.

With $k = 9.5$, our method shows that no reduction is

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*This example is taken from [38]. The simplification is carried out by our method and the response of the simplified system is compared with those obtained from approximated systems given in [38] .

possible. However, Meier and Luenberger [38] have approximated the above system by a second order system given by:

$$\frac{X(s)}{U(s)} = \frac{0.0042\ (s-19)}{(s+1)\ (s+3)} \tag{5.4.13}$$

Reference [38] also gives another dominant-pole approximation to (5.4.10) as:

$$\frac{\overline{X}(s)}{U(s)} = \frac{0.08}{(s+1)(s+3)} \qquad (5.4.14)^*$$

In order to compare the results of our method with (5.4.13) and (5.4.14), we purposely have chosen a smaller value ($k = 7.5$) for k and carried out the simplification process. We get

$$\begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2^* & -a_1^* \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ p \end{bmatrix} u(t)$$

$$\text{and} \quad x^* = 4x_1^* + x_2^* \qquad \ldots(5.4.15)$$

where

$$a_1^* = 2.16302280$$

$$a_2^* = 1.49842202$$

$$\text{and} \quad p = 0.00998948$$

The eigenvalues of the simplified system are $-1.08151139 \pm j\,0.57337231$. The damping constant of this system is approximately 0.89.

---

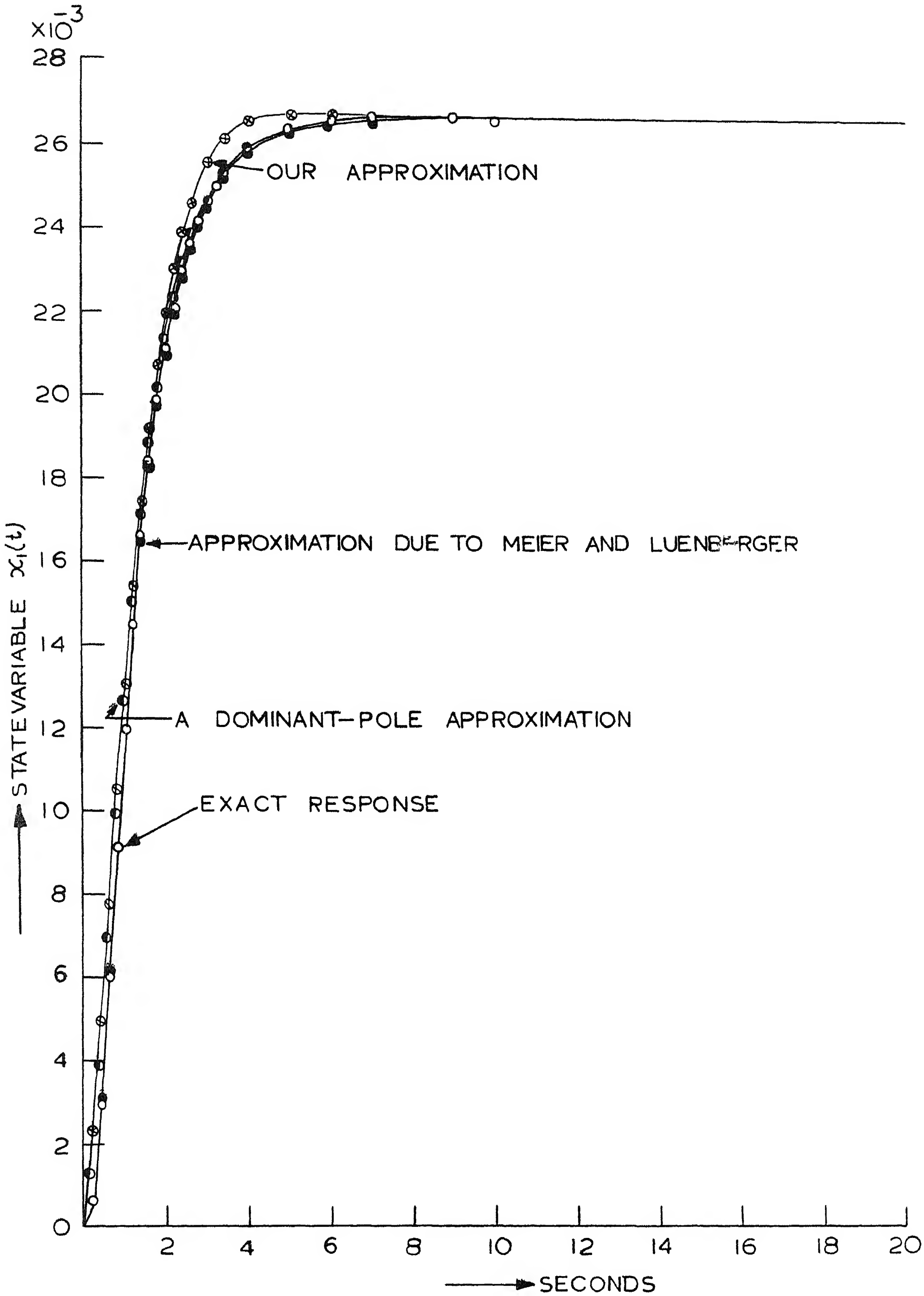
*This is obtained by neglecting s in $(s+4)$, $(s+5)$ and $(s+10)$ appearing in (5.4.10).

The responses $\hat{x}(t)$ (obtained by Meier and Luenberger's method), $\bar{x}(t)$ (a dominant-pole approximation of $x(t)$) and $x^*(t)$ (obtained by our method) for step input are plotted in Figure (5.4.3).

It is to be observed that Meier and Luenberger assume some knowledge about the input function and have to solve a set of nonlinear equations to obtain the simplified system in (5.4.13). This approach may become very difficult if the actual system is of large size.

5.5 EXAMPLE SYSTEMS WITH REAL AND COMPLEX EIGENVALUES

In this section, typical example systems with real and complex eigenvalues will be considered and some problems associated with simplification of such systems will also be discussed. Let us repeat again the following: Whereas the proposed method does not require the knowledge of the eigenvalues of the actual system, just to gain confidence in our method and to gather support for our method, we start with system whose eigenvalues are known. Having clearly followed the fact as to why we often talk about the nature of eivenvalues, let us understand that there are two distinct cases in complex eigenvalue systems. In the first case, the actual system has some complex eigenvalues to be retained in the simplified system. In the second case, the actual system has some "far-off complex eigenvalues" which are to be eliminated in arriving at the simplified system. In each case, several examples



EXACT AND APPROXIMATE RESPONSES FOR EXAMPLE (5.4.6)

FIGURE (5 4 3)

will now be presented.

Example (5.5.1): Consider a system with $\lambda_1 = -100$ and $\lambda_2, \lambda_3 = -0.5 \pm j\ 7.75$. As expected our method results in one successful comparison and the reduced system is found to have eigenvalues very close to λ_2 and λ_3.

Example (5.5.2): Consider a system with $\lambda_1 = -100$, $\lambda_2 = -80$ and $\lambda_3, \lambda_4 = -0.5 \pm j\ 7.75$. One may expect that reduction by order two may be possible in this case. But our method shows that only the first comparison is successful. The reduced system (of order 3) has eigenvalues approximately equal to $-0.5 \pm j\ 7.75$ and -44. It may be argued that as the real eigenvalue is only about 15 times (magnitude wise) the complex eigenvalue, further reduction is not possible.

Example (5.5.3): Consider a system with

$$\lambda_1 = -1, \ \lambda_2, \lambda_3 = -0.5 \pm j\ 2.5, \lambda_4 = -20 \text{ and } \lambda_5 = -30.$$

It is found that

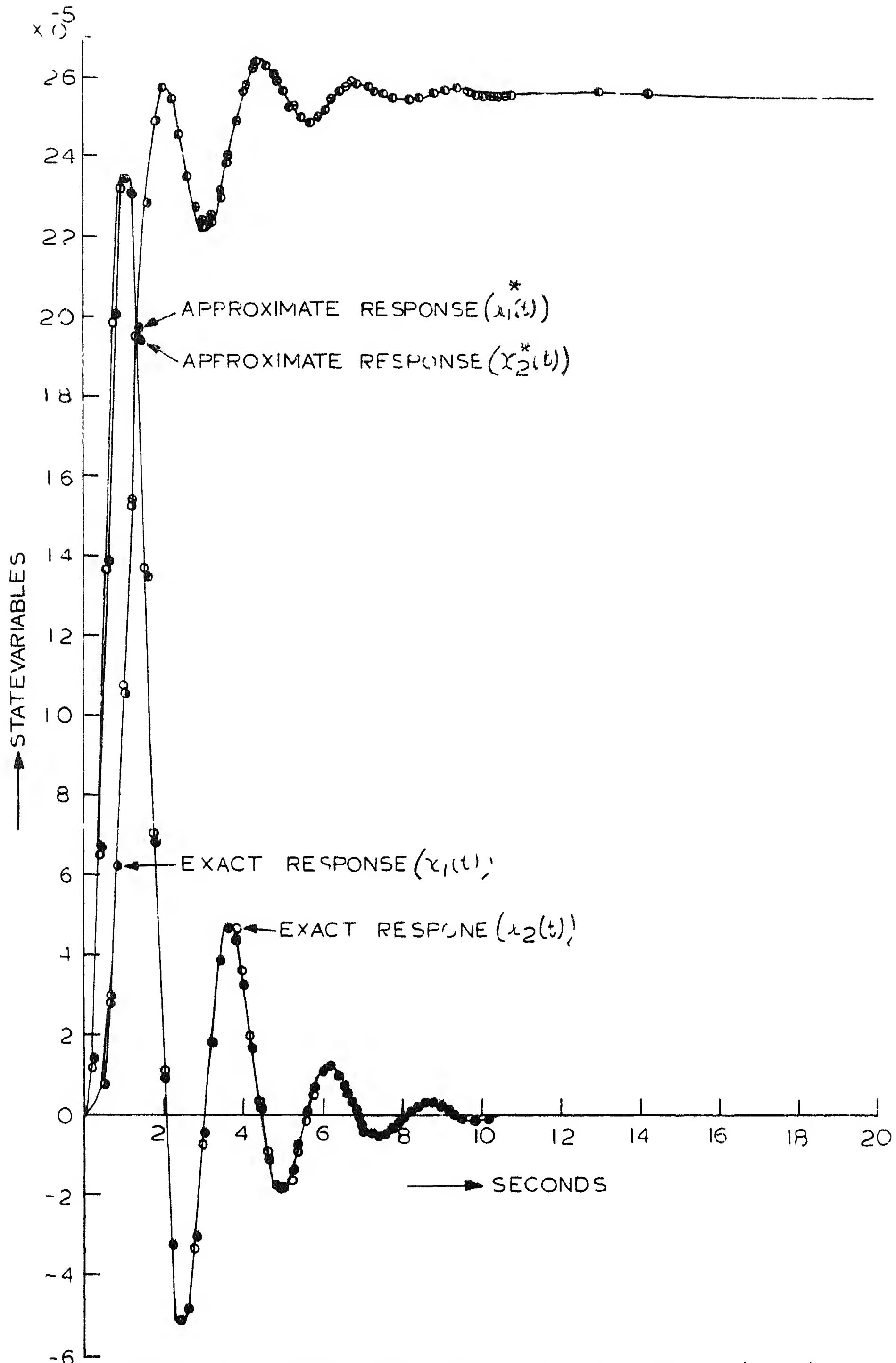
$$b_4/b_3 \cong 29\ , \quad b_3/b_2 \cong 6.15 \quad \text{and} \quad b_2/b_1 \cong 1.18.$$

Thus, the simplified system has the order equal to 4. The simplified system has eigenvalues given by:

$$\lambda_1^* = -1.0311605\ , \qquad \lambda_2^* = -10.9478410$$

$$\text{and} \quad \lambda_3^*, \lambda_4^* = -0.5209461 \pm j\ 2.5243308.$$

The responses of actual and simplified systems are plotted in Figure (5.5.1).



EXACT AND APPROXIMATE RESPONSE FOR EXAMPLE (5 5 3)

FIGURE (5 5 1)

In the above example systems, we had dominant complex eigenvalues.

Example (5.5.4): Consider a system with

$$\lambda_1 = -0.1, \ \lambda_2 = -0.2 \ \text{and} \ \lambda_3, \lambda_4 = -5 \pm j\,5.$$

The simplified system has order two and it has

$$\lambda_1^* = -0.10341686 \quad \text{and} \quad \lambda_2^* = -0.18759833.$$

Thus the eliminated modes correspond to $-5 \pm j\,5$, as expected. Refer to Figure (5.5.2) for responses of the actual and the simplified systems.

Example (5.5.5): Consider a system with

$$\lambda_1 = -0.5, \ \lambda_2 = -1.0 \ \text{and} \ \lambda_3, \lambda_4 = -1.0 \pm j\,25.0.$$

Note that λ_3 and λ_4 are magnitude wise much larger than λ_1 and λ_2. Even though the first comparison is not successful, the second one is and thus the system size can be reduced by two. The simplified system has

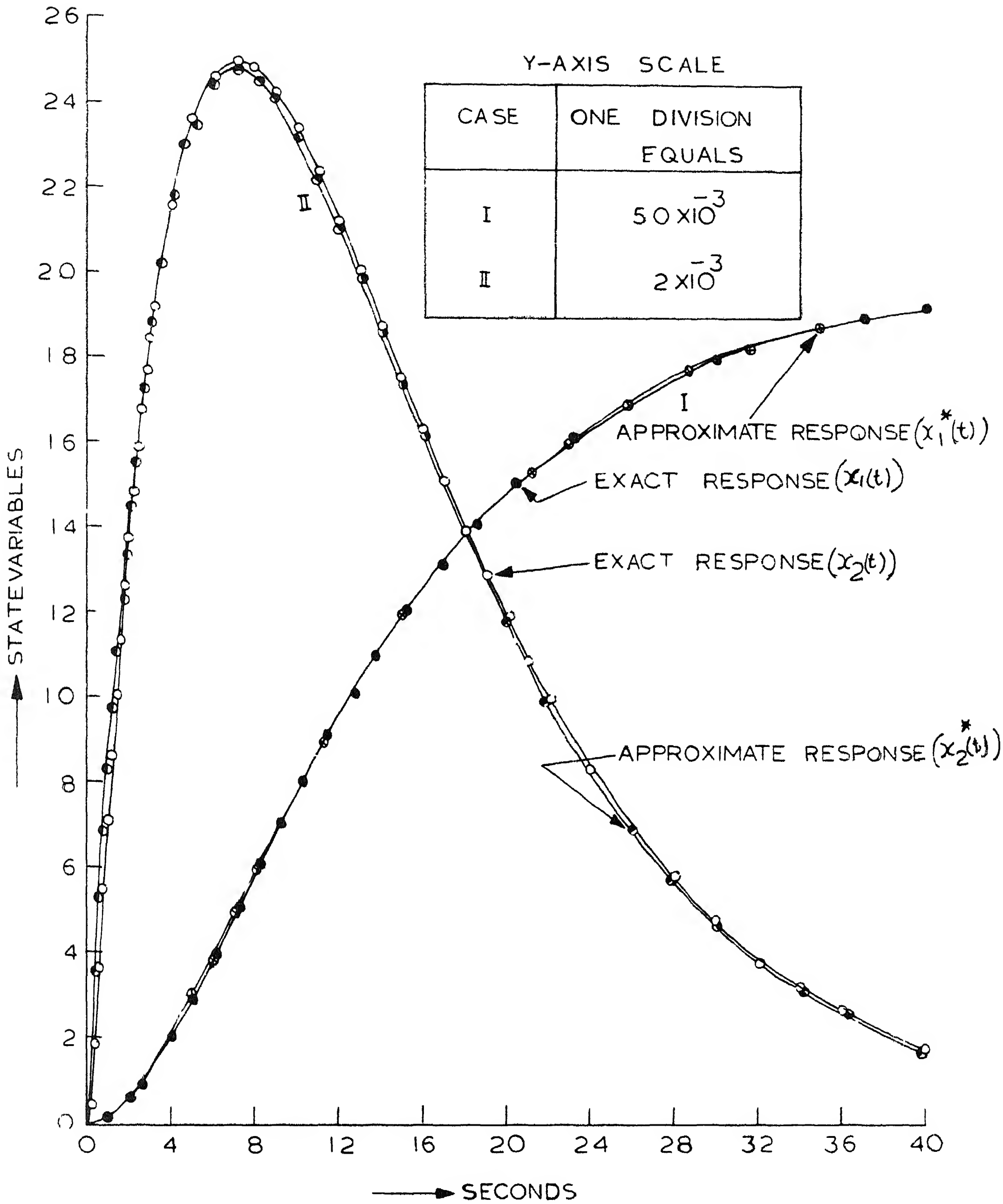
$$\lambda_1^* = -0.39379313 \quad \text{and} \quad \lambda_2^* = -2.20219074.$$

Figure (5.5.3) shows the responses of the actual and simplified systems.

Example (5.5.6): Consider a system with

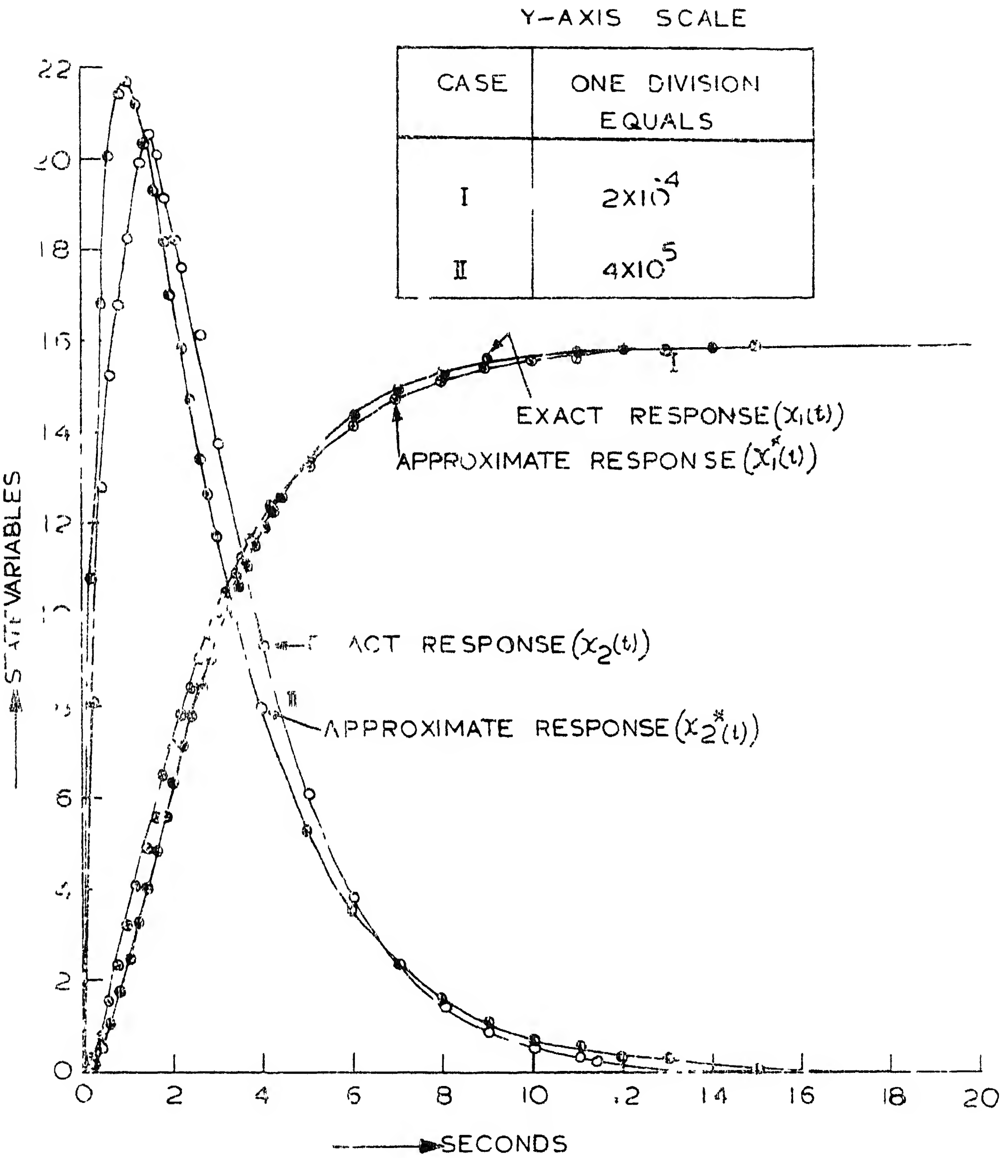
$$\lambda_1 = -1.0, \ \lambda_2, \lambda_3 = -0.5 \pm j\,3.0 \ \text{and} \ \lambda_4, \lambda_5 = -0.05 \pm j\,0.2.$$

The reduced system has dimension 3 and its eigenvalues are:



EXACT AND APPROXIMATE RESPONSES FOR EXAMPLE (5 5 4)

FIGURE (5 5 2)



EXACT AND APPROXIMATE RESPONSE FOR EXAMPLE (5 5 5)

FIGURE(5 5 3)

$\lambda_1^* = -1.6834226$ and $\lambda_2^*, \lambda_3^* = -0.05021498 \pm j\,0.19872369.$

Responses of actual and approximate systems are given in Figure (5.5.4).

So far in almost all the examples, our method has given expected and reasonable results. Let us consider two more examples which present some conceptual difficulties.

Example (5.5.7): Consider a system with

$\lambda_1 = -1.0,\ \lambda_2, \lambda_3 = -0.5 \pm j\,3.0$ and $\lambda_4, \lambda_5 = -8.0 \pm j\,0.2.$

Note that λ_4 and λ_5 are, magnitude wise, not much larger than λ_2 and λ_3. However, our method results in

$$b_4/b_3 = 11,\quad b_3/b_2 = 1.78 \quad\text{and}\quad b_2/b_1 = 1.13.$$

Thus our method says that the given system can be simplified and the simplified system has order 4. The eigenvalues of the simplified system are found to be:

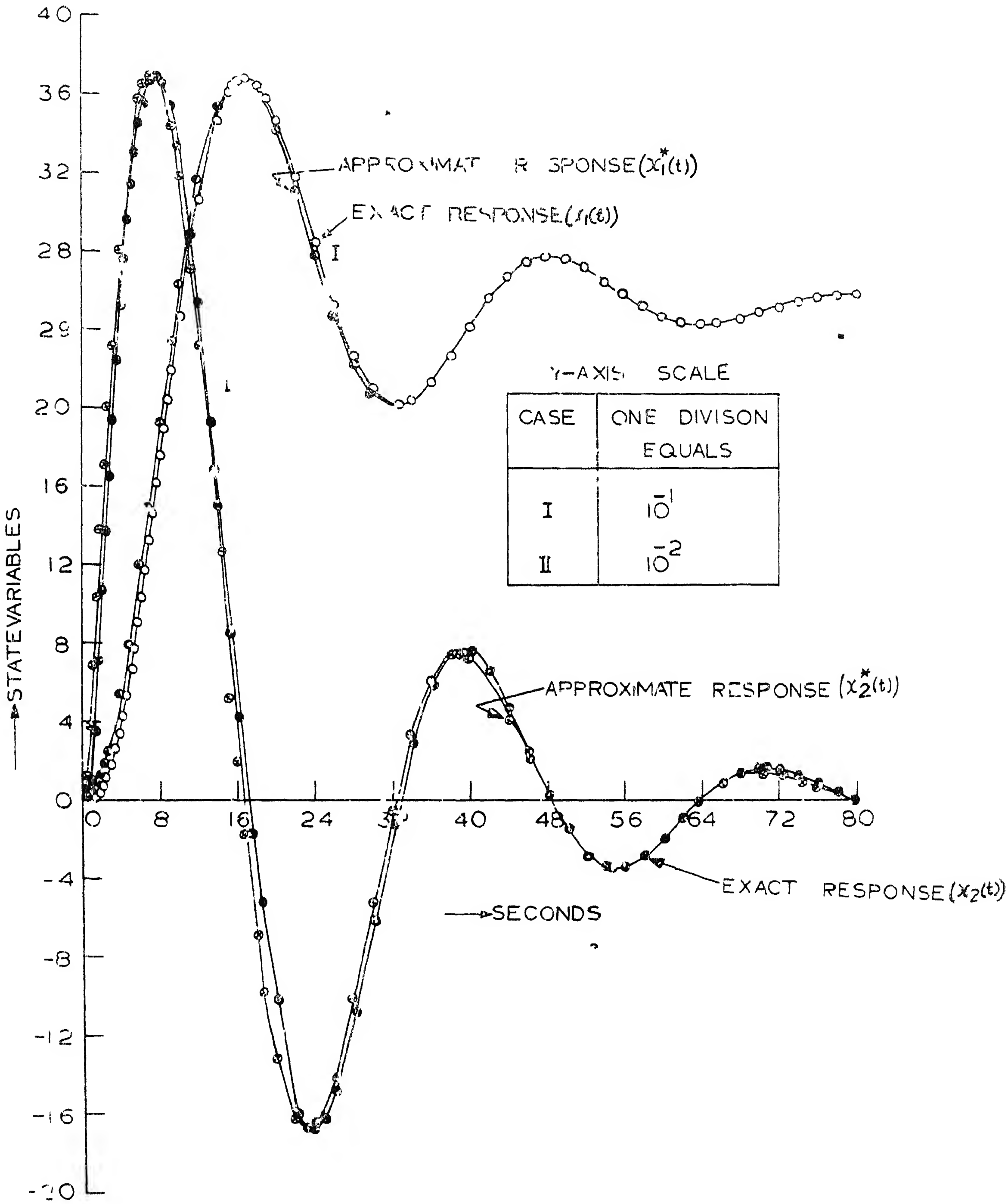
$$\lambda_1^* = -1.1467420\ ,\ \lambda_2^* = -2.9441188$$

and

$$\lambda_3^*, \lambda_4^* = -0.44205414 \pm j\,3.09066888.$$

Note that λ_1^* is close to λ_1 and λ_3^* and λ_4^* are close to λ_2 and λ_3. Thus the following comments are now in order:

1. Even though the 'magnitude criterion' (which is found to be successful in all the previous examples) suggests



EXACT AND APPROXIMATE RESPONSES FOR EXAMPLE (5 5 6)

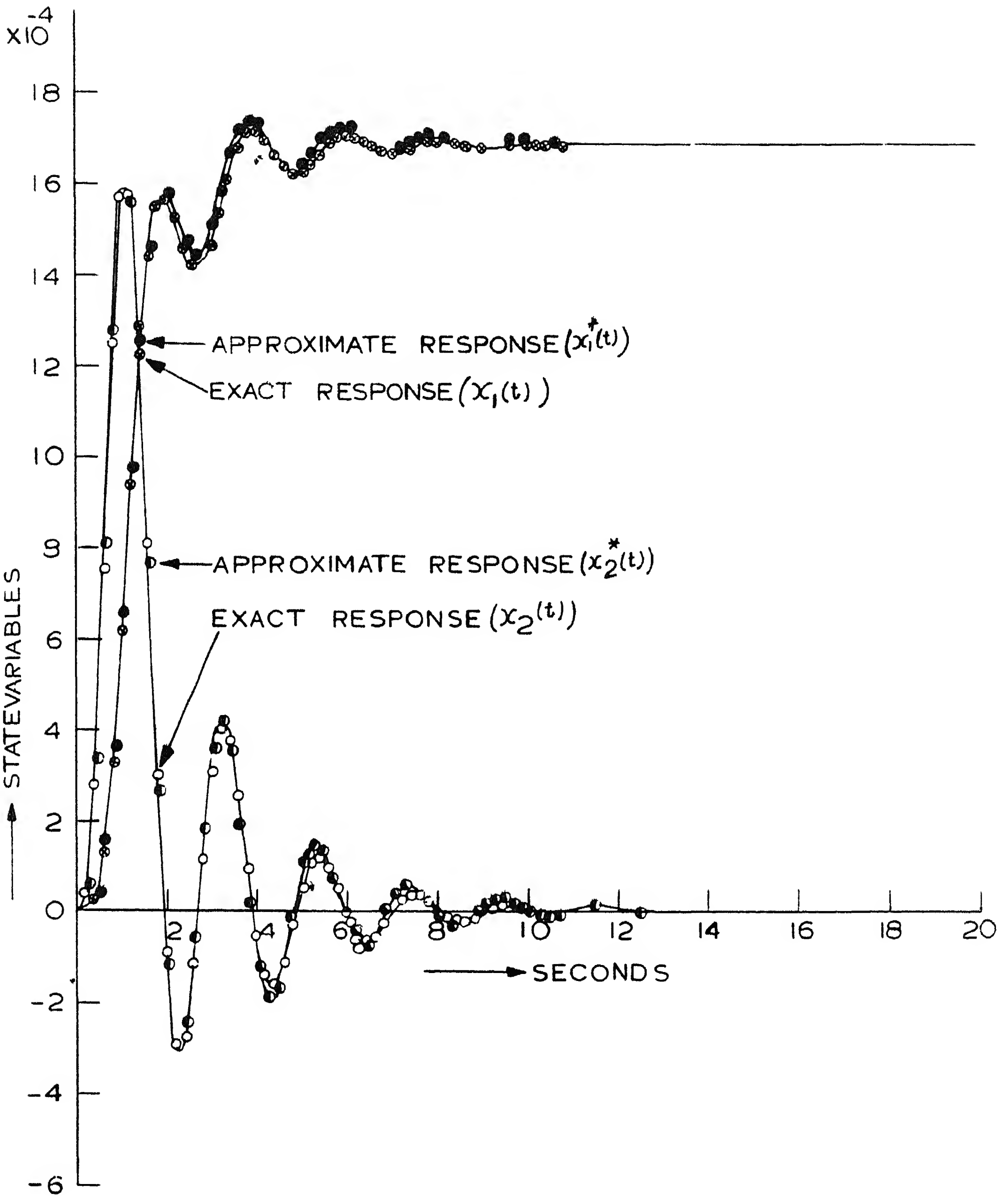
FIGURE (5 5 4)

that no reduction is possible in this case, our method does indicate the possibility of reducing the system by one order.

2. In the simplified system, one eigenvalue in the complex pair λ_4 and λ_5 is eliminated and the other modified drastically.

However, the responses shown in Figure (5.5.5) show that the approximation made is fairly good. It may be argued that the effect of the eigenvalues λ_4 and λ_5 (This pair has a damping constant very close to 1 and can be thus regarded as approximately equivalent to a real repeated eigenvalue at -8.0) is approximately realized by having a single eigenvalue at -2.9441188. A rigorous justification of this statement needs further investigation.

It is interesting to ask as to what will happen if the damping constant δ associated with λ_4 and λ_5 is decreased from its present value. In all cases with $\delta = 0.85, 0.80, 0.707$ and 0.6, it is found that only one comparison is successful thus resulting in the same situation. When $\delta = 0.15$ (the imaginary part of λ_4 and $\lambda_5 \cong 50$), our method indicates that the size of the system can be reduced by two. Observe that in this case, the magnitude of λ_4 or λ_5 is much larger than those of λ_1, λ_2 and λ_3.



EXACT AND APPROXIMATE RESPONSE FOR EXAMPLE (5 5 7)

FIGURE (5 5 5)

Example (5.5.8). Consider a system with

$$\lambda_1, \lambda_2 = -0.5 \pm j\,3 \quad \text{and} \quad \lambda_3, \lambda_4 = -8.0 \pm j\,0.2.$$

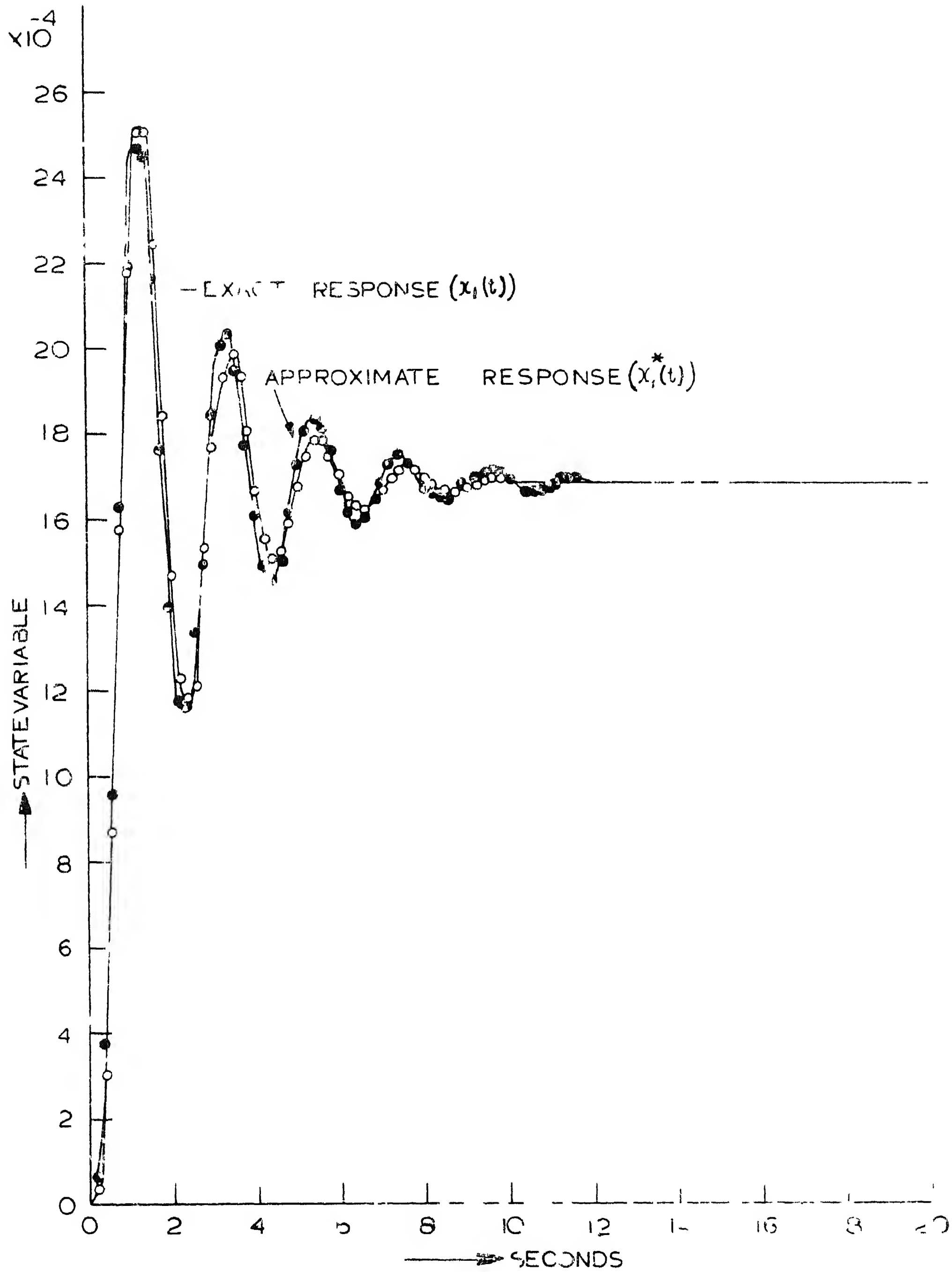
This also resulted in one successful comparison and the simplified system has

$$\lambda_1^* = -3.6951548 \quad \text{and} \quad \lambda_2^*, \lambda_3^* = -0.4117478 \pm j\,3.04310048.$$

All the comments given in Example (5.5.7) may be made for this Example also. Responses of actual and simplified systems are given in Figure (5.5.6).

Before ending this section, let us observe the following: From all the examples presented in section 5.4 and 5.5, it is inferred that given any system described as in (5.2.3), without worrying about the eigenvalues, the system may be simplified by the proposed method and that the simplified system response is "close" to the actual system response. However, the error associated with derivative variables (x_2^*, x_3^*, etc.) is considerably large compared to that associated with the variable x_1^*.

Results of the typical examples considered in sections 5.4 and 5.5 are summarized in Table (5.5.1). The various responses for actual and approximate systems for these examples were computed using the program "Modified Digital Analog Simulator" written for IBM 7044. For simulation of the system in Schwarz form, refer to section 2.3.



EXACT AND APPROXIMATE RESPONSES FOR EXAMPLE (5 5 8)

FIGURE(5 5 6)

TABLE (5.5.1)

ACTUAL AND APPROXIMATE SYSTEMS DATA

Example No.	Actual System Data: Order	Actual System Data: Schwarz elements	Actual System Data: Eigen-values	Simplified System Data: Order	Simplified System Data: Schwarz elements	Simplified System Data: Eigen-values
(5.4.1)	6	0.03590879	-0.2	3	0.03590879	-0.16883593
		0.27054737	-0.4		0.27054737	-0.35450246 ± j0.24782361
		3.18777084	-0.5		0.87736430	
		35.24095232	-5.0			
		350.14481408	-10.0			
		36.09999968	-20.0			
(5.4.2)	4	1.83335350	-1.0	2	1.83335350	-1.12617126
		72.92392704	-2.0		2.75412392	-1.62795264
		2727.24269056	-50.0			
		103.00000000	-50.0			
(5.4.6)	4	1.50000000	-1.0	2	1.50000000	-1.08151139 ± j0.57337231
		11.40000000	-3.0		2.16302280	
		100.10000000	-5.0			
		19.00000000	-10.0			
(5.5.3)	5	3.20538708	-1.0	4	3.20538708	-1.03116130
		3.80996440	-0.5±j 2.5		3.80996440	-10.94784100
		23.39810976	-20.0		23.39810976	-0.52094610 ± j2.52433080
		677.08653056	-30.0		13.02089472	
		52.00000000				

Table (5.5.1) (Contd.)

(5.5.4)	4	0.01940083	-0.1	2	0.01940083	-0.10341686
		1.45632678	-0.2		0.29101519	-0.18759833
		51.54427072	-5.0±j 5.0			
		10.29999992				
(5.5.5)	4	0.86720758	-0.5	2	0.86720758	-0.38379313
		267.70421504	-1.0		2.59598388	-2.20219074
		360.92857344	-1.0±j25.0			
		3.50000000				
(5.5.6)	5	0.03964259	-1.0	3	0.03964259	-1.68342260
		0.17143613	-0.5±j 3.0		0.17143613	-0.05021498 ± j0.19872369
		4.72225436	-0.05±j0.2		1.78385260	
		5.55916672				
		2.09999996				
(5.5.7)	5	4.05029576	-1.0	4	4.05029576	-1.14674200
		4.56506204	-0.5±j3.0		4.56506204	-2.94411880
		8.12519726	-8.0±j0.2		8.12519736	-0.44205414 ± j3.09066888
		89.54944384			4.97499080	
		18.00000000				
(5.5.8)	4	7.71143768	-0.5±j 3.0	3	7.71143768	-3.69515480
		4.76150336	-8.0±j 0.2		4.76150336	-0.41174780 ± j3.04310048
		76.81705856			4.51865044	
		17.00000000				

NOTE:

The Schwarz elements stated in the above Table are in the order $b_1, b_2 \ldots, b_n$ for actual system and $b_1^*, b_2^*, \ldots, b_{n^*}^*$ for the simplified system where n and n^* represent respectively the order of the actual system and that of the simplified system. Except for Example (5.4.6), $k = 9.5$ was used (See section 5.4) for all Examples and for Example (5.4.6), $k = 7.5$ was used.

5.6 PROBLEMS FOR FUTURE INVESTIGATION

A few problems associated with the proposed simplification method may be investigated further. The first one is a systematic extension of this method to overcome some of the difficulties encountered with complex eigenvalue systems. Second, the method may be suitably modified to bring multivariable systems also within its range of application. In this connection, the Schwarz canonical form for multivariable systems and its network interpretation discussed in chapter IV may be useful. Third, procedures may be developed to estimate the amount of error between the exact and approximate responses. Finally, it may be possible to determine a performance index which becomes optimum for the proposed simplification method.

5.7 CONCLUSION

The proposed method has the major advantage that the system eigenvalues and eigenvectors are not to be evaluated. Schwarz canonical form has been used both for determining whether the system can be simplified or not and for simplifying the system. Schwarz form is also useful in detecting redundant state variables and eliminating them. (See Appendix V).

The simplified system has a few spectral energy density moments exactly same as those of the actual systems.

For details, refer to [42, 25] .

Several examples have been provided to illustrate the simplification method. However, it is to be emphasized that as is common with many simplification procedures, the proposed method also does not give reasonable results for all inputs.

CHAPTER - VI

A CANONICAL FORM FOR LINEAR TIME-INVARIANT DISCRETE-TIME SYSTEMS

6.1 INTRODUCTION

The Schwarz canonical form for linear time-invariant continuous systems has mainly the following three properties: (1) stability of the system is inferred by inspection of the elements of the canonical matrix; (2) Liapunov function for the system is constructed quite simply and (3) the elements of the canonical matrix can be computed from the elements of the first column of the Routh array. It is the aim of this chapter to develop- quite analogous to the development in the continuous case- a canonical form for linear time-invariant discrete-time systems. In fact, the canonical form proposed in this chapter has the first two properties stated above. Analogous to the aforementioned property (3), attempt is made to compute the elements of the proposed canonical matrix from those of the Jury table. Presently this attempt is not completely successful. However, it is hoped that in future such a link- between the proposed canonical matrix and the Jury table- would be established.

Contents in this chapter will now be rapidly stated. Section 2 systematically derives the canonical form. It also discusses some properties of the canonical matrix. Use of this canonical form in investigating the system*

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*For convenience, instead of every time saying 'linear time-invariant discrete-time system', the word 'system' is used throughout this chapter.

stability as well as in constructing a Liapunov function for the system is discussed in section 3. Section 4 gives the relationships between the elements of the canonical matrix. The fairly lengthy section 5 enlists a few methods of computing the elements of the canonical matrix. Finally, a few problems are posed for further investigation in section 6.

6.2 PROPOSED CANONICAL FORM

The treatment given in this section closely parallels [45]. The concept of matrix bilinear transformation [45, 47] is freely used in the sequel. To make matters simple in the actual derivation of the canonical form, the properties of the matrix bilinear transformation will first be enunciated without proofs (For proofs, refer to [45]).

$\underline{V}$ is defined as the matrix bilinear transformation of $\underline{U}$ if

$$\underline{V} = (\underline{U} + \underline{I})(\underline{U} - \underline{I})^{-1} \qquad (6.2.1)$$

Note carefully that the above definition presumes that $(\underline{U} - \underline{I})^{-1}$ exists. In other words, the matrix $\underline{U}$ does not have an eigenvalue (or multiple eigenvalues) at 1*. The statement of the properties of the matrix bilinear transformation is now in order.

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*Throughout the discussion in this chapter, it is assumed that the discrete system under investigation has no eigenvalues at 1. For a stable system, this is automatically satisfied, as a stable system, if continuous, has all eigenvalues in the left half-plane and if discrete, within the unit circle.

1. $(\underline{U} + \underline{I})$ and $(\underline{U} - \underline{I})^{-1}$ commute. Thus

$$\underline{V} = (\underline{U} + \underline{I})(\underline{U} - \underline{I})^{-1} = (\underline{U} - \underline{I})^{-1}(\underline{U} + \underline{I}) \quad (6.2.2)$$

2. A matrix commutes with its bilinear transform. That is,

$$\underline{U}\,\underline{V} = \underline{V}\,\underline{U} \quad (6.2.3)$$

3. The bilinear transformation possesses the property of reciprocity. This means that if $\underline{V}$ is the bilinear transform of $\underline{U}$, then $\underline{U}$ is the bilinear transform of $\underline{V}$. That is, given (6.2.1),

$$\underline{U} = (\underline{V} + \underline{I})(\underline{V} - \underline{I})^{-1} \quad (6.2.4)$$

4. If

$$\underline{U} = \underline{S}\,\underline{T}\,\underline{S}^{-1}, \quad (6.2.5)$$

then (6.2.1) implies that

$$\underline{V} = \underline{S}(\underline{T} + \underline{I})(\underline{T} - \underline{I})^{-1}\underline{S}^{-1} \quad (6.2.6)$$

5. If all eigenvalues of a matrix are in the left half plane (unit circle), those of its bilinear transform are in the unit circle (left half plane).

6. A matrix and its bilinear transform have the same eigenvectors. Further, the bilinear transform of a non-derogatory (derogatory) matrix is itself non-derogatory (derogatory)*.

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*A matrix is said to be non-derogatory, if it has only one eigenvector per eigenvalue and derogatory, otherwise. If a matrix is non-derogatory, it is similar to its companion form (phase-variable form).

The equation (6.2.1) can be put in a more useful form as under·

$$\underline{V} = \underline{I} + 2(\underline{U} - \underline{I})^{-1} \qquad (6.2.7)$$

Before proceeding to the derivation of the canonical form, a clear statement as to what is aimed at may prove to be useful. Given any stable system (discrete) described by

$$\underline{x}_{k+1} = \underline{A}\,\underline{x}_k \qquad (6.2.8)$$

where $\underline{A}$ is non-derogatory. We wish to transform (6.2.8) to a canonical form

$$\underline{y}_{k+1} = \underline{W}\,\underline{y}_k \qquad (6.2.9)$$

where $\underline{W}$ is similar to $\underline{A}$. In what follows, we consider $\underline{W}$ as the bilinear transform of the Schwarz matrix $\underline{B}$ of the continuous system obtained from (6.2.8) through the bilinear transformation. In the next section, we shall see what the stability of (6.2.8) or (6.2.9) implies in terms of the elements of $\underline{W}$.

Define $\underline{C}_s$ as the companion form of any matrix (non-derogatory) $\underline{S}$. We are now ready for deriving the canonical form. As $\underline{A}$ is non-derogatory,

$$\underline{C}_a = \underline{N}\,\underline{A}\,\underline{N}^{-1} \qquad (6.2.10)$$

Let $\underline{D}$ be the bilinear transform of $\underline{C}_a$. Then by the property 4 stated above,

$$\underline{D} = \underline{N}\,(\underline{A} + \underline{I})\,(\underline{A} - \underline{I})^{-1}\,\underline{N}^{-1} \qquad (6.2.11)$$

By property 6, it is clear that $\underline{D}$ is also non-derogatory. Therefore,

$$\underline{C}_d = \underline{M}\,\underline{D}\,\underline{M}^{-1}$$

i.e.

$$\underline{C}_d = \underline{M}\,\underline{N}\,(\underline{A} + \underline{I})\,(\underline{A} - \underline{I})^{-1}\,\underline{N}^{-1}\,\underline{M}^{-1} \qquad (6.2.12)$$

Transform $\underline{C}_d$ to its Schwarz form $\underline{B}$. That is,

$$\underline{B} = \underline{P}\,\underline{C}_d\,\underline{P}^{-1} \qquad (6.2.13)$$

or

$$\underline{B} = (\underline{P}\,\underline{M}\,\underline{N})\,(\underline{A} + \underline{I})\,(\underline{A} - \underline{I})^{-1}\,(\underline{P}\,\underline{M}\,\underline{N})^{-1}$$

Denote

$$\underline{S} = \underline{P}\,\underline{M}\,\underline{N} \qquad (6.2.14)$$

Then

$$\underline{B} = \underline{S}\,(\underline{\Lambda} + \underline{I})\,(\underline{\Lambda} - \underline{I})^{-1}\,\underline{S}^{-1}$$

Invoking the properties 3 and 4 ,

$$(\underline{B} + \underline{I})\,(\underline{B} - \underline{I})^{-1} = \underline{S}\,\underline{A}\,\underline{S}^{-1}$$

Define

$$\underline{W} = (\underline{B} + \underline{I})\,(\underline{B} - \underline{I})^{-1} \qquad (6.2.15)$$

Thus

$$\underline{W} = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1} \qquad (6.2.16)$$

Thus we have shown that $\underline{W}$ as defined in (6.2.15) is similar to $\underline{\Lambda}$.

Now a few comments are in order. Let $\underline{G}$ be the bilinear transform of $\underline{A}$. Thus,

$$\underline{G} = (\underline{A} + \underline{I})\ (\underline{A} - \underline{I})^{-1} \tag{6.2.17}$$

Transform $\underline{G}$ to its Schwarz form $\underline{B}$.

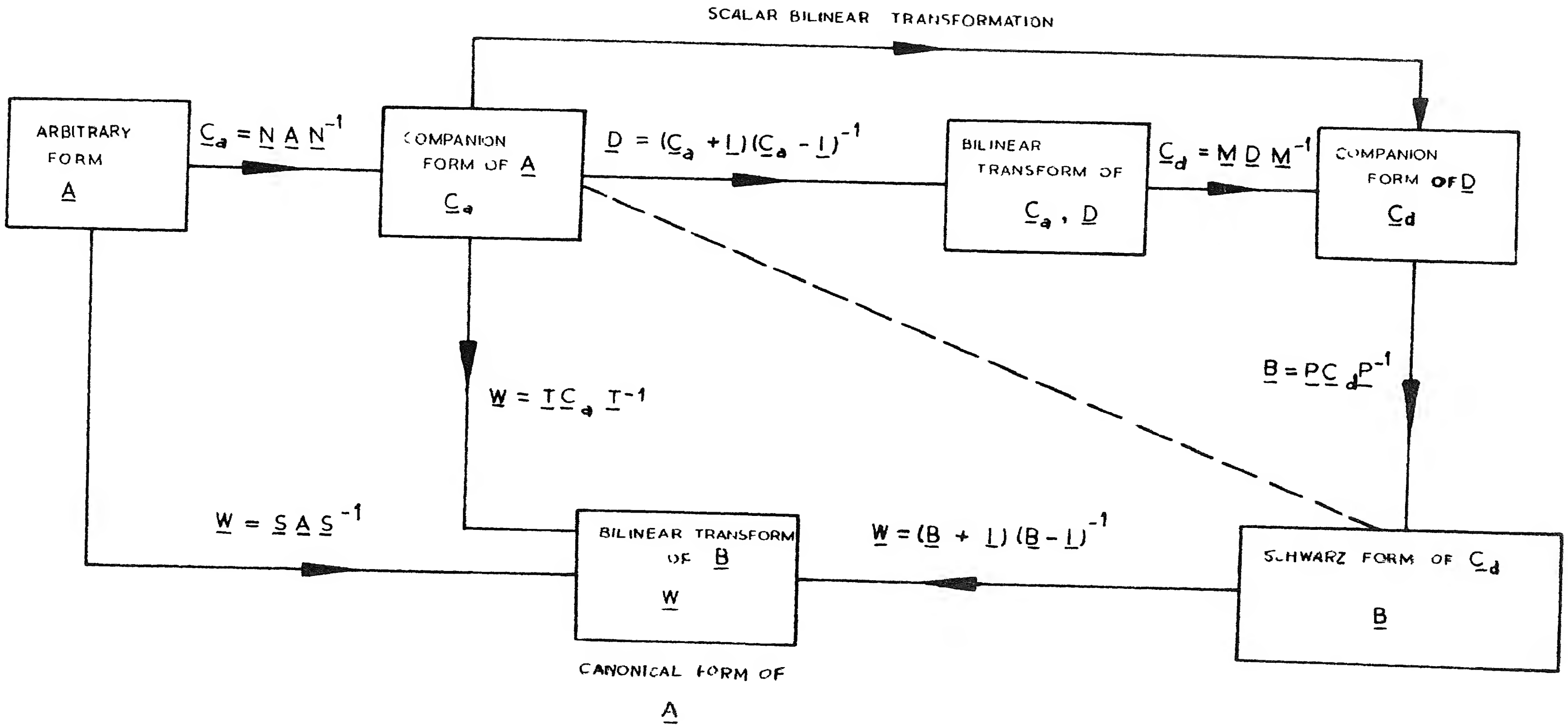
$$\underline{B} = \underline{H}^{-1}\ \underline{G}\ \underline{H} = \underline{H}^{-1}\ (\underline{A} + \underline{I})\ (\underline{A} - \underline{I})^{-1}\ \underline{H} \tag{6.2.18}$$

Therefore, from (6.2.15)

$$\underline{W} = \underline{H}^{-1}\ \underline{A}\ \underline{H} \tag{6.2.19}$$

Note that in equations (6.2.17) through (6.2.19), transformation to companion form is not involved. One may wonder whether non-derogatoriness restriction can be relaxed or not. It cannot be relaxed because $\underline{H}^{-1}$ exists only when $\underline{G}$ (hence $\underline{A}$, by property 6) is non-derogatory. However, a problem where $\underline{A}$ is derogatory can be tackled. For details refer to [45] .

The several steps in the derivation of the canonical matrix $\underline{W}$ may be well illustrated with the help of the Figure (6.2.1). The scalar bilinear transformation $(z = \frac{s+1}{s-1})$ which is used to find the elements of $\underline{C}_d$ from those of $\underline{C}_a$ becomes very tedious when the order of the system becomes large. However, an improved technique which has been recently proposed by Power [47] may be used to achieve the same purpose. A method of computing $\underline{D}$ from $\underline{C}_a$ is given in section 6.5. The transformation



TRANSFORMATION TO THE PROPOSED CANONICAL FORM

FIGURE (6 2 1)

matrix $\underline{P}$ which transforms $\underline{C}_d$ to $\underline{B}$ is discussed in chapter III. The elements w_{ij}'s of $\underline{W}$ may be computed from the elements b_i's of $\underline{B}$ in two ways. Method 1 uses a set of rules which help in computing w_{ij}'s from b_i's and this is discussed in section 6.3. Method 2, which makes use of recursive relations between w_{ij}'s and b_i's is presented in section 6,5. In section 6.5, attempts are made to express b_i's directly in terms of the elements of $\underline{C}_a$ (say a_i's) and to relate w_{ij}'s with a_i's through the elements of the Jury table of the characteristic polynomial associated with $\underline{C}_a$.

Some interesting properties of the canonical matrix $\underline{W}$ (for a stable system) are stated below:

1. $$\underset{i>j}{w_{ij}} = w_{ji}\left[\prod_{k=j}^{i-1} (-b_k)\right] \tag{6.2.20}$$

or

$$\underset{i>j}{w_{ij}} = (-1)^{i-j}\, w_{ji}\left[\prod_{k=j}^{i-1} b_k\right] \tag{6.2.21}$$

2. $$\underset{j>i}{w_{ij}} < 0 \tag{6.2.22}$$

3. $$w_{(s+i)s} = \begin{cases} < 0 & i \text{ even} \\ > 0 & i \text{ odd} \end{cases} \tag{6.2.23}$$

$$(i=1,2,\ldots,n-1)$$

$$(s=1,2,\ldots,n-i)$$

Properties 2 and 3 will now be explained in words. All the superdiagonal elements of the canonical matrix are negative. Elements of the canonical matrix lying in odd-ordered subdiagonals are positive and those lying in even-ordered subdiagonals are negative. While dealing with many numerical examples, it is found that

$$\langle \underline{w}_i \,,\ \underline{w}_j \rangle \ < \ 1 \ , i \neq j \qquad (6.2.24)$$

where $\underline{w}_i$ and $\underline{w}_j$ are respectively the i-th and the j-th columns of $\underline{W}$. Proof of (6.2.24) in general is not available presently. However, in what follows proofs of (6.2.20), (6.2.22) and (6.2.23) are given.

Proof of (6.2.20):

The commutative property given in equation (6.2.3) of the matrix bilinear transformation is used below in proving (6.2.20) by induction. From (6.2.3),

$$\underline{W}\,\underline{B} = \underline{B}\,\underline{W} \qquad (6.2.25)$$

Equating the corresponding elements of $\underline{W}\,\underline{B}$ and $\underline{B}\,\underline{W}$ on the leading diagonal gives relations between elements of $\underline{W}$ on the first sub- and superdiagonals; equating corresponding elements of $\underline{W}\,\underline{B}$ and $\underline{B}\,\underline{W}$ on the first sub- and superdiagonals gives relations between elements of W on the second sub- and superdiagonals; and so on.

$$(\underline{W}\,\underline{B})_{jj} = (\underline{B}\,\underline{W})_{jj}$$

gives

$$w_{j(j-1)} - b_j\, w_{j(j+1)} = -b_{j-1}\, w_{(j-1)j} + w_{(j+1)j}, \quad j \geq 2 \qquad \ldots \quad (6.2.26)$$

(6.2.26) implies

$$w_{(j+1)j} = -b_j\, w_{j(j+1)}, \qquad (6.2.27)$$

provided

$$w_{j(j-1)} = -b_{j-1}\, w_{(j-1)j} \qquad (6.2.28)$$

Now, $(\underline{W}\ \underline{B})_{11} = (\underline{B}\ \underline{W})_{11}$ results in $w_{21} = -b_1 w_{12}$. Thus from (6.2.27), $w_{32} = -b_2\, w_{23}$ and thus $w_{43} = -b_3\, w_{32}$ and so on. So by induction it has been shown that

$$w_{(j+1)j} = -b_j\, w_{j(j+1)} \quad , (j=1,2,\ldots,n-1) \qquad \ldots \quad (6.2.29)$$

Next, from

$$(\underline{W}\ \underline{B})_{j(j-1)} = (\underline{B}\ \underline{W})_{j(j-1)}$$

we get

$$w_{j(j-2)} - w_{(j+1)(j-1)} = -b_{j-1} w_{(j-1)(j-1)} + b_{j-1} w_{jj} \quad , j > 2 \qquad \ldots \quad (6.2.30)$$

Similarly

$$(\underline{W}\ \underline{B})_{(j-1)j} = (\underline{B}\ \underline{W})_{(j-1)j}$$

yields

$$w_{(j-1)(j-1)} - b_j\, w_{(j-1)(j+1)} = -b_{j-2}\, w_{(j-2)j} + w_{jj}, \quad j > 2 \qquad \ldots \quad (6.2.31)$$

Multiplying (6.2.31) by b_{j-1} and rearranging result in

$$-b_j\, b_{j-1}\, w_{(j-1)(j+1)} + b_{j-1}\, b_{j-2}\, w_{(j-2)j} = -b_{j-1}\, w_{(j-1)(j-1)} + b_{j-1}\, w_{jj} \qquad (6.2.32)$$

Note that the right hand sides of (6.2.30) and (6.2.32) are identical and therefore we have:

$$w_{(j+1)(j-1)} = b_j\, b_{j-1}\, w_{(j-1)(j+1)} \qquad (6.2.33)$$

provided

$$w_{j(j-2)} = b_{j-1}\, b_{j-2}\, w_{(j-2)j},\ j > 2 \qquad (6.2.34)$$

But from $(\underline{W}\ \underline{B})_{21} = (\underline{B}\ \underline{W})_{21}$ and $(\underline{W}\ \underline{B})_{12} = (\underline{B}\ \underline{W})_{12}$, after some manipulation, we get

$$w_{31} = b_1 b_2 w_{13}.$$

Therefore from (6.2.33) and (6.2.34), $w_{42} = b_2 b_3 w_{24}$ which in turn implies $w_{53} = b_3 b_4 w_{35}$ and so on. So, by induction it has been shown that

$$w_{(j+1)(j-1)} = (-b_j)(-b_{j-1})\, w_{(j-1)(j+1)},\ (j=2,3,\ldots,n-1) \qquad \ldots (6.2.35)$$

The same procedure may be continued for the second sub- and superdiagonal elements, third sub- and superdiagonal elements and so on to obtain the equations

$$w_{(s+i)s} = (-b_i)(-b_{i+1})\dots(-b_{s+i-1})\, w_{i(s+i)}$$
$$i=1,2,\dots,n-1$$
$$s=1,2,\dots,n-i \qquad (6.2.36)$$

(6.2.36) can be rewritten as in (6.2.20). This completes the proof.

Proof of (6.2.22).

Define $\underline{F} = (\underline{B} - \underline{I})^{-1}$. Then

$$\underline{F}\,(\underline{B} - \underline{I}) = \underline{I} \qquad (6.2.37)$$

(6.2.37) implies $[\underline{F}(\underline{B} - \underline{I})]_{in} = 0$, $i < n$. That is,

$$f_{i(n-1)} - (1 + b_n)\, f_{in} = 0$$

or

$$f_{i(n-1)} = (1 + b_n)\, f_{in} \qquad (6.2.38)$$

Equating $[\underline{F}(\underline{B} - \underline{I})]_{i(n-1)}$ to zero, we get

$$f_{i(n-2)} = f_{i(n-1)} + b_{n-1}\, f_{in} \qquad (6.2.39)$$

Repeating this procedure, we get

$$f_{i(n-k)} = f_{i(n-k+1)} + b_{(n-k+1)}\, f_{i(n-k+2)}, \quad (k=2,3,\dots,n-i-1) \qquad \dots (6.2.40)$$

From the above equations (6.2.38) - (6.2.40), it is clear that all the superdiagonal elements lying in the i-th row of $\underline{F}$ have the same sign as that of f_{in}. As $\underline{W}$ is the

matrix bilinear transform of $\underline{B}$, from (6.2.7) we get

$$\underline{W} = \underline{I} + 2(\underline{B} - \underline{I})^{-1} \tag{6.2.41}$$

Therefore w_{ij} $(j > i)$ $= 2 f_{ij}$. Thus proving (6.2.22) reduces to showing that $f_{ij} < 0$ $(j > i)$ and hence it has to be shown that $f_{in} < 0$.

Referring to Appendix I Equations (A.1.3) ,

$$f_{in} = (1/B(n))\ F(j-1)\ (-1)^{j+n} \tag{6.2.42}$$

As $B(n)$ has the sign of $(-1)^n$ and $F(j-1)$ that of $(-1)^{j-1}$, it is readily seen from (6.2.42) that $f_{in} < 0$.

<u>Proof of (6.2.23)</u>:

This property of $\underline{W}$ given in (6.2.23) is obtained by combining (6.2.21) and (6.2.22). From (6.2.21),

$$w_{(s+i)s} = (-1)^s\ w_{s(s+i)} \left[\sum_{k=s}^{(s+i-1)} b_k \right] \tag{6.2.43}$$

From (6.2.22), $w_{s(s+i)} < 0$. Therefore, for a stable system (b_i's > 0), the sign of $w_{(s+i)s}$ is that of $(-1)^{s+1}$. Hence the relations (6.2.23).

6.3 STABILITY INVESTIGATION

In this section, first, simple rules to compute the elements of $\underline{W}$ from those of $\underline{B}$ will be stated. Second, the proposed canonical form is used in constructing Liapunov functions for the system. In section 6.2, a stable

<u>Rule I</u> <u>To evaluate Δ_n</u>

$$\Delta_n = 1 + \sum_{i=1}^{n} b_i + \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} b_i b_j + \sum_{i=1}^{n-4} \sum_{j=i+2}^{n-2} \sum_{k=j+2}^{n} b_i b_j b_k + \cdots\cdots \qquad \ldots (6.3.2)$$

For example, with $n = 5$, the above rule gives

$$\Delta_5 = 1 + b_1 + b_2 + b_3 + b_4 + b_5 + b_1 b_3 + b_1 b_4 + b_1 b_5 + b_2 b_4 + b_2 b_5 + b_3 b_5 + b_1 b_3 b_5.$$

We observe that Δ_n can also be found recursively by using the relations (Appendix I).

$$\left.\begin{aligned} B(0) &= 1 \\ B(1) &= -(1 + b_n) \\ B(k) &= -B(k-1) + b_{n-k+1} B(k-2), \; k = 2,3,\ldots,n \end{aligned}\right\} \qquad (6.3.3)$$

Then

$$\Delta_n = B(n)$$

However, we require Δ_n to be written explicitly as in (6.3.2) for applying the other rules to be stated.

<u>Rule II:</u> <u>To determine the diagonal elements of W</u>

From (6.3.1),

$$w_{ii} = \frac{w'_{ii}}{\Delta_n}$$

Replace b_1 by $(-b_1)$ in the expression for $(-\Delta_n)$

to get w'_{11}. Replace b_2 by $(-b_2)$ in w'_{11} to get w'_{22}. To determine w'_{kk} $(k = 3,4,\ldots,n)$, replace b_{k-2} and b_k by $(-b_{k-2})$ and $(-b_k)$ respectively in $w'_{(k-1)(k-1)}$.

<u>Rule III</u> <u>To determine the superdiagonal elements of W</u>

$w'_{i(i+1)}$ $(i=1,2,\ldots,n-1)$ is obtained by replacing b_{i+1}, b_i and b_{i-1} (when $i=1$, these are b_2 and b_1 only) by zeros in the expression for w'_{ii} and multiplying the result by two. Then, any w'_{ij}, $j > (i+1)$, $i=1,2,\ldots,n$, $j = i+2,(i+3),\ldots,n$ is found out by just replacing b_j by zero in the expression for $w'_{i(j-1)}$.

<u>Rule IV:</u> <u>To determine the subdiagonal elements of $\tilde{W}$</u>

$$w_{ij}\ (i > j) = w_{ji} \prod_{k=j}^{i-1} (-b_k)$$

A few comments are now in order. First, for applying the rules II and III, we must explicitly write Δ_n in terms of b_i's as shown in (6.3.2). Second, these rules are quite simple both for remembering and for applying. Finally, it is possible to derive relations which help us to algorithmically determine w_{ij}'s from the knowledge of b_i's. This will be done in section 6.5. Let us now illustrate the application of the above rules by means of a simple example.

<u>Example</u>: Given

$$b_1 = 1, \quad b_2 = 2, \quad b_3 = 3 \quad \text{and} \quad b_4 = 4.$$

To compute $\underline{W}$.

Now, applying the above rules, we obtain

$$\hat{\Delta}_4 = 1 + b_1 + b_2 + b_3 + b_4 + b_1b_3 + b_1b_4 + b_2b_4 = 26$$

$$w'_{11} = -1 + b_1 - b_2 - b_3 - b_4 + b_1b_3 + b_1b_4 - b_2b_4 = -10$$

$$w'_{22} = -1 + b_1 + b_2 - b_3 - b_4 + b_1b_3 + b_1b_4 + b_2b_4 = 10$$

$$w'_{33} = -1 - b_1 + b_2 + b_3 - b_4 + b_1b_3 - b_1b_4 + b_2b_4 = 6$$

$$w'_{44} = -1 - b_1 - b_2 + b_3 + b_4 + b_1b_3 + b_1b_4 + b_2b_4 = 18$$

$$w'_{12} = 2(-1 - b_3 - b_4) = -16$$

$$w'_{23} = 2(-1 - b_4) = -10$$

$$w'_{34} = 2(-1 - b_1) = -4$$

$$w'_{13} = 2(-1 - b_4) = -10$$

$$w'_{14} = 2(-1) = -2$$

$$w'_{24} = 2(-1) = -2$$

$$w'_{21} = -b_1\, w'_{12} = 16$$

$$w'_{31} = b_2b_1\, w'_{13} = -20$$

$$w'_{32} = -b_2\, w'_{23} = 20$$

$$w'_{41} = -b_1b_2b_3\, w'_{14} = 12$$

$$w'_{42} = b_2b_3\, w'_{24} = -12$$

$$w'_{43} = -b_3\, w'_{34} = 12$$

Thus

$$\underline{W} = \frac{1}{26}\begin{bmatrix} -10 & -16 & -10 & -2 \\ 16 & 10 & -10 & -2 \\ -20 & 20 & 6 & -4 \\ 12 & -12 & 12 & 18 \end{bmatrix}$$

Observe that all the superdiagonal elements of $\underline{W}$ are negative. That all elements on the first and the third subdiagonals are positive and that those on the second subdiagonal are negative. Also

$$\langle \underline{w}_1 , \underline{w}_2 \rangle = - \frac{224}{26x26} < 1$$

$$\langle \underline{w}_1 , \underline{w}_3 \rangle = - \frac{36}{26x26} < 1$$

$$\langle \underline{w}_1 , \underline{w}_4 \rangle = \frac{284}{26x26} < 1$$

$$\langle \underline{w}_2 , \underline{w}_3 \rangle = \frac{36}{26x26} < 1$$

$$\langle \underline{w}_2 , \underline{w}_4 \rangle = - \frac{284}{26x26} < 1$$

$$\langle \underline{w}_3 , \underline{w}_4 \rangle = \frac{232}{26x26} < 1$$

Thus we see that $\langle \underline{w}_i , \underline{w}_j \rangle < 1$, $i \neq j$. In fact, when $i = j = 1$, this inner product becomes $900/676 > 1$. Let us next consider the problem of constructing Liapunov functions for the system.

Consider the discrete system described by (6.2.8).

Transform the system through

$$\underline{y}_k = \underline{S}\ \underline{x}_k \tag{6.3.4}$$

so that the equation (6.2.9) represents the transformed system. Let the Liapunov function for the canonical system in (6.2.9) be given by:

$$V_k = \langle \underline{y}_k , \underline{L}\ \underline{y}_k \rangle \tag{6.3.5}$$

where the form of $\underline{L}$ will be chosen later. The Liapunov function V_k expressed in terms of the original system state vector becomes

$$V_k = \langle \underline{x}_k , \underline{S}^T \underline{L}\ \underline{S}\ \underline{x}_k \rangle \tag{6.3.6}$$

The first difference of the Liapunov function may be readily written as

$$V_{k+1} - V_k = \langle \underline{y}_k , (\underline{W}^T \underline{L}\ \underline{W} - \underline{L})\ \underline{y}_k \rangle \tag{6.3.7}$$

or

$$V_{k+1} - V_k = \langle \underline{x}_k , \underline{S}^T (\underline{W}^T \underline{L}\ \underline{W} - \underline{L})\ \underline{S}\ \underline{x}_k \rangle \tag{6.3.8}$$

We note that $\underline{L}$ in (6.3.5) has to be appropriately chosen such that the first difference of the Liapunov function given in (6.3.7) becomes negative definite (or negative semidefinite with the additional qualification that $V_{k+1} - V_k$ does not vanish along any solution trajectory of (6.2.9) except the origin). Choose tentatively,

$$\underline{L} = \text{diag.} \left\{ (b_1 b_2 \ldots b_n),\ (b_2 b_3 \ldots b_n), \ldots, b_{n-1} b_n,\ b_n \right\} \qquad \ldots (6.3.9)$$

Denote

$$\underline{W}^T \underline{L}\, \underline{W} - \underline{L} = \underline{Q} = \left[q_{ij} \right] \qquad (6.3.10)$$

Then we can write that

$$V_{k+1} - V_k = - \left[(-q_{11})^{\frac{1}{2}} y_{k1} - (-q_{22})^{\frac{1}{2}} y_{k2} + (-q_{33})^{\frac{1}{2}} y_{k3} - (-q_{44})^{\frac{1}{2}} y_{k4} + \ldots\ldots \right]^2 \qquad \ldots (6.3.11)$$

As will be seen, q_{11} , q_{22} etc. are negative and thus the square roots of $(-q_{11})$, $(-q_{22})$ etc. are all real numbers. The expression in (6.3.11) is negative semidefinite.

It may be verified that

$$q_{ii} = - \frac{1}{\Delta_n^2} \left\{ 2 b_n b_{n-1} \ldots b_i (1 + b_{i-2} + b_{i-3} + \cdots + b_1) \right\}^2$$

$$i = 1, 2, \ldots, n$$

$$b_k = 0 \text{ if } k \leqslant 0 \qquad (6.3.12)$$

$$q_{ij} = (-1)^{(i+j+1)} \left\{ (-q_{ii})^{\frac{1}{2}} (-q_{jj})^{\frac{1}{2}} \right\}$$

$$j > i \qquad (i = 1, 2, \ldots, n-1)$$

$$(j = i+1\ ,\ i+2, \ldots, n) \qquad (6.3.13)$$

$$q_{ij} = q_{ji}\ , \quad (j = 1, 2, \ldots, n-1),\ (i = j+1,\ j+2, \ldots, n)$$

$$i > j \qquad \ldots (6.3.14)$$

It can be shown that

$$q_{k2} = -b_1 \; q_{k1} \; , \; (k = 1,2,\ldots,n) \qquad (6.3.15)$$

Owing to (6.3.15), only the first principal minor of $\underline{Q}$ is different from zero. This principal minor, q_{11}, is negative in view of (6.3.12). Therefore, $\underline{Q}$ is negative semidefinite. To illustrate whatever has been said so far, let us consider a simple example.

Example: For $n = 3$, construct $\underline{Q}$ and $\underline{y}_k^T \, \underline{Q} \, \underline{y}_k$.

Applying the formulas given in (6.3.12) - (6.3.14),

$$Q = \frac{1}{\Delta_3^2} \begin{bmatrix} -(2b_1b_2b_3)^2 & (2b_1b_2b_3)(2b_2b_3) & -(2b_1b_2b_3)\{2b_3(1+b_1)\} \\ (2b_1b_2b_3)(2b_2b_3) & -(2b_2b_3)^2 & (2b_2b_3)\{2b_3(1+b_1)\} \\ -(2b_1b_2b_3)\{2b_3 \; (1+b_1)\} & (2b_2b_3)\{2b_3(1+b_1)\} & -\{2b_3(1+b_1)\}^2 \end{bmatrix}$$

where $\Delta_3 = 1 + b_1 + b_2 + b_3 + b_1b_3$

Note that $q_{21} = -b_1 \, q_{22}$ and $q_{31} = -b_1q_{32}$. Now, from (6.3.11) and (6.3.12)

$$V_{k+1} - V_k = \underline{y}_k^T \; \underline{Q} \, \underline{y}_k$$

$$= - \left\{ \frac{2b_1b_2b_3}{\Delta 3} \, y_{k1} - \frac{2b_2b_3}{\Delta 3} \, y_{k2} + \frac{2b_3(1+b_1)}{\Delta 3} \, y_{k3} \right\}^2$$

It is yet to be concluded that the negative semidefiniteness of $\underline{Q}$ assures the asymptotic stability of the system in (6.2.9). Following reference [46], it may be possible to give a general proof for the fact that the first difference $V_{k+1} - V_k$ is not identically zero for any general sequence of vectors. However, in this section, a simple example will be considered to illustrate this idea.

<u>Example</u>: Consider a system with $n = 3$. Let $b_1 = 2$, $b_2 = 3$ and $b_3 = 4$. The system in the canonical form can be written as

$$\begin{bmatrix} y_{(k+1)1} \\ y_{(k+1)2} \\ y_{(k+1)3} \end{bmatrix} = \begin{bmatrix} 2/18 & -10/18 & -2/18 \\ 20/18 & 8/18 & -2/18 \\ -12/18 & 6/18 & 12/18 \end{bmatrix} \begin{bmatrix} y_{k1} \\ y_{k2} \\ y_{k3} \end{bmatrix}$$

And,

$$V_{k+1} - V_k = -\left\{ \frac{48}{18} y_{k1} - \frac{24}{18} y_{k2} + \frac{24}{18} y_{k3} \right\}^2$$

The vector $\underline{\delta}$ (see reference [46]) is given by

$$\underline{\delta}^T = \left(\frac{48}{18}, -\frac{24}{18}, \frac{24}{18} \right)$$

As in [46], let us form a matrix $\underline{D}$ as

$$\underline{D} = \begin{bmatrix} \underline{\delta}^T \\ \underline{\delta}^T \underline{W} \\ \underline{\delta}^T \underline{W}^2 \end{bmatrix}$$

That is

$$\underline{D} = \begin{bmatrix} 24/9 & -12/9 & 12/9 \\ -56/27 & -44/27 & 12/27 \\ -568/243 & -140/243 & 172/243 \end{bmatrix}$$

We find that $|\underline{D}| \neq 0$. So following [46], we conclude that $v_{k+1} - V_k$ is not identically zero for any general sequence of vectors.

It may be of interest to express $\underline{L}$ and $\underline{Q}$ in terms of w_{ij}'s themselves. From the relation (6.2.20), the following may be readily obtained:

$$\prod_{k=1}^{m} b_k = (-1)^m \frac{w_{(m+1)1}}{w_{1(m+1)}}, \quad m \leq (n-1) \tag{6.3.16}$$

The elements of the diagonal matrix $\underline{L}$ are given by

$$\ell_{ii} = b_n b_{n-1} \ldots b_i$$

or

$$\ell_{ii} = \frac{b_n b_{n-1} \ldots b_2 b_1}{b_1 b_2 b_3 \ldots b_{i-1}} \tag{6.3.17}$$

From (6.3.17) and (6.3.16) we get

$$\ell_{ii} = (-1)^{n-i} b_n \frac{w_{n1}}{w_{1n}} \frac{w_{1i}}{w_{i1}} \tag{6.3.18}$$

Using (6.3.12), (6.3.16) and (6.2.20), we can obtain

$$q_{kk} = -\left[2(-1)^{n-i} b_n \frac{w_{n1}}{w_{1n}} \frac{w_{1k}}{w_{k1}} \left\{1 - \frac{w_{21}}{w_{12}} - \frac{w_{31}}{w_{13}} \frac{w_{12}}{w_{21}} - \frac{w_{41}}{w_{14}} \frac{w_{13}}{w_{31}} - \ldots - \frac{w_{(k-1)1}}{w_{1(k-1)}} \frac{w_{1(k-2)}}{w_{(k-2)1}}\right\}\right]^2 \ldots \tag{6.3.19}$$

From the definition of $\underline{W}$, it may be easily seen that

$$\underline{W}\ \underline{B} = \underline{W} + \underline{B} + \underline{I} \tag{6.3.20}$$

Equating the nn element on both sides, we get after rearrangement

$$b_n = \frac{1 - w_{n(n-1)} + w_{nn}}{1 - w_{nn}} \tag{6.3.21}$$

Thus equations (6.3.18) - (6.3.21) with (6.3.13), (6.3.14) and (6.3.11) determine $\underline{L}$, $\underline{Q}$ and $\underline{y}_k^T\ \underline{Q}\ \underline{y}_k$ in terms of the elements of $\underline{W}$.

Referring to (6.3.6), we note that to construct the Liapunov function for the system in (6.2.8), knowledge of $\underline{S}$, the transformation matrix is required. From (6.2.14),

$$\underline{S} = \underline{P}\ \underline{M}\ \underline{N}$$

The determination of $\underline{M}$ and $\underline{N}$ are discussed in [20] . For determining $\underline{M}$, $\underline{D}$ given in (6.2.11) has to be first evaluated. The transformation matrix $\underline{P}$ has been discussed in chapter III. Reference [20] has given a method to compute a number of transformation matrices $\underline{M}$ and $\underline{N}$. Hence a number of Liapunov functions V_k can be constructed.

Suppose $\underline{W}$ has been computed somehow. In this case, instead of finding $\underline{P}$, $\underline{M}$ and $\underline{N}$ in order to evaluate $\underline{S}$, another method is possible. If

$$\bar{\underline{C}}_a = \underline{N}\ \underline{A}\ \underline{N}^{-1}$$

a transformation $\underline{T}$ can be found [20] such that

$$\underline{C}_a = \underline{T}\ \bar{\underline{W}}\ \bar{\underline{T}}^{-1}$$

Then

$$\underline{S} = \underline{T}^{-1}\ \bar{\underline{N}} \tag{6.3.22}$$

Finally, let us derive a set of conditions on the elements of $\underline{W}$ for the asymptotic stability of the system (6.2.8) or (6.2.9). It is known (chapter II) that the system (discrete) is asymptotically stable if all ℓ_{ii}'s > 0 $(i=1,2,\ldots,n)$. From (6.3.16)

$$\frac{w_{n1}}{w_{1n}} = \left(\frac{b_1 b_2 \ldots b_n}{b_n}\right)(-1)^{n-1} \tag{6.3.23}$$

That is

$$b_n(w_{n1}/w_{1n}) = \ell_{11}\ (-1)^{n-1} \tag{6.3.24}$$

Now $\ell_{ii} > 0$ implies that

$$(-1)^{n-1}\ b_n \frac{w_{n1}}{w_{1n}} \frac{w_{1i}}{w_{i1}} > 0$$

From (6.3.24) this in turn implies that

$$(-1)^{i+1}\ \ell_{11}\ (w_{1i}/w_{i1}) > 0$$

As $\ell_{11} > 0$ (we want this to be so), the above gives

$$(-1)^{i+1} \, (w_{1i}/w_{i1}) > 0 \; , \; i=2,3,\ldots,n \quad (6.3.25)$$

From (6.3.24)

$$\ell_{11} = (-1)^{n+1} \, b_n \, (w_{n1}/w_{1n}) \qquad (6.3.26)$$

As we have forced [by (6.3.25)] $(-1)^{n+1} \, (w_{1n}/w_{n1}) > 0$, ℓ_{11} in (6.3.26) > 0 if $b_n > 0$. That is, [see equation (6.3.21)]

$$\frac{1 - w_{n(n-1)} + w_{nn}}{1 - w_{nn}} > 0 \qquad (6.3.27)$$

Equations (6.3.25) and (6.3.27) represent the conditions on the elements of $\underline{W}$ for the asymptotic stability of the system.

6.4 RELATIONS BETWEEN THE ELEMENTS OF $\underline{W}$

From the results of the previous section, it may be observed that the n^2 elements of $\underline{W}$ are uniquely determined in terms of the n elements b_i $(i=1,2,\ldots,n)$. So it is reasonable to expect that only n of the n^2 elements w_{ij} $(i,j = 1,2,\ldots,n)$ are independent and that the other elements are related to these "independent" elements. Already we have seen that the subdiagonal elements are related to the superdiagonal ones through b_i's as in (6.2.20). In this section, we are going to state the relations between subdiagonal elements and

between superdiagonal elements of $\underline{W}$. Relations between the elements on the tri-diagonals of $\underline{W}$ (leading, first sub- and first super-diagonals) will also be given. Finally it is illustrated that the knowledge of the first row elements of $\underline{W}$ only will suffice for calculating the rest of the elements of $\underline{W}$. A simple example is also provided at the end of this section.

The main equations used for arriving at the several expressions to be given later are:

$$\underline{W}\ \underline{B} = \underline{W} + \underline{B} + \underline{I} \tag{6.4.1}$$

$$\underline{B}\ \underline{W} = \underline{B} + \underline{W} + \underline{I} \tag{6.4.2}$$

The derivation of the results of this section involves straightforward (though tedious) procedure of equating the corresponding elements on both sides of (6.4.1) as well as those of (6.4.2). Hence, we will state the results only. Of course, the equations used for obtaining such results will be mentioned.

First, we shall write down the relations obtained by equating the first row and the n-th row elements of $(\underline{B}\ \underline{W})$ and $(\underline{W} + \underline{B} + \underline{I})$.

$$\left.\begin{aligned} w_{21} &= 1 + w_{11} \\ w_{22} &= 1 + w_{12} \\ w_{2k} &= w_{1k}\ ,\ k = 3,4,\ldots,n \end{aligned}\right\} \tag{6.4.3}$$

$$\left.\begin{aligned} b_{n-1} &= -(1+b_n)\frac{w_{nk}}{w_{(n-1)k}}, \quad k = 1,2,\ldots,n-2 \\ b_{n-1} &= \frac{w_{n(n-1)}\,(1+b_n)}{1-w_{(n-1)(n-1)}} \\ b_{n-1} &= \frac{1+w_{nn}+b_n\,(w_{nn}-1)}{w_{(n-1)n}} \end{aligned}\right\} \qquad (6.4.4)$$

Equating the elements on the tridiagonals of both sides of (6.4.2) results in

$$b_k = \frac{w_{(k+1)k} - w_{(k+2)k}}{1-w_{kk}}, \quad (k = 1,2,\ldots,n-2) \qquad (6.4.5)$$

$$b_k = \frac{w_{(k+2)(k+1)} - w_{(k+1)(k+1)} - 1}{w_{k(k+1)}}, \quad (k=1,2,\ldots,n-2) \qquad \ldots (6.4.6)$$

$$b_k = \frac{w_{(k+2)(k+2)} - w_{(k+1)(k+2)} - 1}{w_{k(k+2)}}, \quad (k=1,2,\ldots,n-2) \qquad \ldots (6.4.7)$$

Note that equations (6.4.5) - (6.4.7) express the relations between the elements on the tridiagonals of $\underline{W}$. We shall now state the relations between the superdiagonal elements of $\underline{J}$.

$$h_k = \frac{w_{(k+2)m} - w_{(k+1)m}}{w_{km}} \qquad (6.4.8)$$

$$(k = 1,2,\ldots,n-3)$$

$$(m = k+3,\ k+4,\ldots,n)$$

(6.4.8) is obtained by equating the elements lying above the first superdiagonal (except those in the first row) of ($\underline{B}\ \underline{W}$) and ($\underline{B} + \underline{W} + \underline{I}$). Relations (6.4.8) can be further manipulated to obtain the following simple and elegant relations:

$$\frac{w_{i(i+1)}}{w_{1(i+1)}} = \frac{w_{i(i+2)}}{w_{1(i+2)}} = \ldots = \frac{w_{in}}{w_{1n}} \qquad (6.4.9)$$

$$i = 2,3,\ldots,n-2$$

Let us explain what (6.4.9) means. The superdiagonal elements of $\underline{W}$ have the interesting property that each element in any i-th row ($i \leq n-1$) bears the same ratio with the element in its own column in the first row as any other superdiagonal element in the same i-th row does with its corresponding first row element. Next let us demonstrate that a similar relation exists among the subdiagonal elements of $\underline{W}$.

Equating the elements below the first subdiagonal of ($\underline{W}\ \underline{B}$) and ($\underline{B} + \underline{W} + \underline{I}$) [see equation (6.4.1)] yields

$$\left.\begin{array}{l} -b_1 = \dfrac{w_{k1}}{w_{k2}}, \quad k = 3,4,\ldots,n \\[2ex] b_k = \dfrac{w_{m(k-1)} - w_{mk}}{w_{m(k+1)}} \\[2ex] k = 2,3,\ldots,n-2 \\ m = k+2,\ k+3,\ldots,n \end{array}\right\} \qquad (6.4.10)$$

Equations (6.4.10) can be further manipulated to obtain the simple relations

$$\frac{w_{(i+1)i}}{w_{(i+1)1}} = \frac{w_{(i+2)i}}{w_{(i+2)1}} = \ldots = \frac{w_{ni}}{w_{n1}} \qquad i = 2,3,\ldots,n-2. \tag{6.4.11}$$

Equation (6.4.11) means that the subdiagonal elements of $\underline{W}$ have the interesting property that each element in any j-th column ($j < n-1$) bears the same ratio with the element in its own row in the first column as any other subdiagonal element in the same j-th column does with its corresponding first column element.

Suppose all the elements in the first row of $\underline{W}$ are known. We can systematically make use of the relations (6.4.3), (6.4.5) - (6.4.9) and (6.2.20) to determine the other elements of $\underline{W}$. Let us illustrate this procedure by a simple example.

Example: For $n=3$, assuming that w_{11}, w_{12} and w_{13} are known, express other elements of $\underline{W}$ in terms of those three.

From (6.4.2) we can obtain

$$\left.\begin{aligned} w_{21} &= 1 + w_{11} \\ w_{22} &= 1 + w_{12} \\ w_{23} &= w_{13} \end{aligned}\right\} \tag{6.4.12}$$

$$\left.\begin{aligned} b_1 &= \frac{w_{21} - w_{31}}{1 - w_{11}} \\ b_1 &= \frac{w_{32} - 1 - w_{22}}{w_{12}} \\ b_1 &= \frac{w_{33} - w_{23} - 1}{w_{13}} \end{aligned}\right\} \qquad (6.4.13)$$

Using (6.2.20)

$$\left.\begin{aligned} b_1 &= -w_{21}/w_{12} \\ b_2 &= -w_{32}/w_{23} \end{aligned}\right\} \qquad (6.4.14)$$

Combining (6.4.13) and (6.4.14) we can write that

$$\left.\begin{aligned} w_{31} &= \frac{w_{21}}{w_{12}} \left(1 - w_{11} + w_{12} \right) \\ w_{32} &= 1 + w_{22} - w_{21} \\ w_{33} &= 1 + w_{23} - \frac{w_{21}}{w_{12}} \; w_{13} \end{aligned}\right\} \qquad (6.4.15)$$

Use (6.4.12) to compute the second row elements of $\underline{W}$. Knowing thus the first two row elements of $\underline{W}$, use (6.4.15) to compute the third row elements of $\underline{W}$. Observe from (6.4.13), for b_1 to be finite $w_{11} \neq 1$, $w_{12} \neq 0$ and $w_{13} \neq 0$. For stable systems, such difficulties do not arise.

6.5 METHODS OF COMPUTING $\underline{W}$ FROM $\underline{A}$

The practical utility of the proposed canonical form depends upon the ease with which elements of $\underline{W}$ may be determined from those of $\underline{A}$. At present the only use of this canonical form is in the construction of Liapunov functions. Is the construction procedure through this canonical form any simpler than the usual method of solving the $n(n+1)/2$ equations

$$\underline{A}^T \underline{G} \underline{A} - \underline{G} = -\underline{Q} \quad ?$$

This is partly answered in this section by presenting systematically a few methods of computing $\underline{W}$. Also attempt has been made to find elements of $\underline{W}$ from the elements of the Jury table. This attempt is not completely successful. However, it may be possible in future to establish the link between the Jury table and the $\underline{W}$ matrix elements and to find some more uses for this canonical form as have been found for the Schwarz canonical form in the continuous case.

Let us consider first the straightforward method which is used while deriving $\underline{W}$ in section 6.2. First transform $\underline{A}$ to its companion form $\underline{C}_a$. To obtain $\underline{C}_d$ from $\underline{C}_a$, either we can apply scalar bilinear transformation, $z = (s+1)/(s-1)$ to the characteristic polynomial $F(z)$ of $\underline{C}_a$ or we can apply the simple rules stated in [47] which involve multiplication of

two similarly formed matrices. The latter approach is found to be simple and elegant. However, to determine the transformation matrix $\underline{M}$ [see equation (6.2.12)] , we require to find $\underline{D}$. Let the characteristic polynomial of $\underline{C}_a$ be given as:

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0 \qquad (6.5.1)$$

Define

$$\left.\begin{aligned} p_0 &= 1 \\ p_1 &= 1 + a_1 \\ p_2 &= 1 + a_1 + a_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ p_n &= 1 + a_1 + a_2 + \cdots + a_n \end{aligned}\right\} \qquad (6.5.2)$$

Then $\underline{D}$ is given by [45] :

$$\underline{D} = \frac{1}{2} p_n (\underline{D}_1 - \underline{D}_2) \qquad (6.5.3)$$

where $\underline{D}_1$ is a lower triangular matrix given by

$$\left.\begin{aligned} (\underline{D}_1)_{ii} &= p_n/2 \\ (\underline{D}_1)_{ij} &= 0 \qquad j > i \\ (\underline{D}_1)_{ij} &= p_n \qquad i > j \end{aligned}\right\} \qquad (6.5.4)$$

and $\underline{D}_2$ is given by

$$(\underline{D}_2)_{ij} = p_{n-j} \; (i,j = 1,2,\ldots,n) \qquad (6.5.5)$$

Having found $\underline{D}$ from (6.5.3), $\underline{M}$ can be obtained using the method discussed in [20]. Note clearly that if the transformation $\underline{S}$ ($= \underline{P}\ \underline{M}\ \underline{N}$) is not required or if $\underline{S}$ is found by using the equation (6.3.22), then determination of $\underline{D}$ is not necessary. In such a case, directly $\underline{C}_d$ can be found from $\underline{C}_a$ as explained earlier. Next form the Routh array of the characteristic polynomial of $\underline{C}_d$. Use the relations between the b_i's and the Routh array elements R_{ij}'s [see chapter II] to compute the b_i's $(i = 1,2,\ldots,n)$. Apply the simple rules (discussed in section 6.3) to determine the elements of $\underline{W}$ knowing the b_i's.

Instead of using the rules given in section 6.3, recursive relations may also be used for computing w_{ij}'s from b_i's. These relations can be derived by simply equating the corresponding elements of $(\underline{W}\ \underline{B})$ and $(\underline{W} + \underline{B} + \underline{I})$ [see equation (6.4.1)].

$$\left.\begin{aligned} w_{12} &= -\frac{1}{b_1}(1 + w_{11}) \\ w_{22} &= 1 + w_{12} \\ w_{21} &= -b_1 w_{12} \end{aligned}\right\} \qquad (6.5.6)$$

To compute the elements on the first superdiagonal:

$$w_{i(i+1)} = -\frac{1}{b_i}\left(w_{ii} + 1 - w_{i(i-1)}\right), \quad (i = 2,3,\ldots,n-1) \qquad (6.5.7)$$

To compute the elements on the second superdiagonal of $\underline{W}$:

$$w_{i(i+2)} = -\frac{1}{b_{i+1}}\left[1 + w_{i(i+1)} - w_{ii}\right] \qquad (i = 1,2,\ldots,n-2) \tag{6.5.8}$$

To compute rest of the superdiagonal elements of $\underline{W}$:

$$w_{ij} = -\frac{1}{b_{j-1}}\left[w_{i(j-1)} - w_{i(j-2)}\right] \qquad \begin{array}{l}(i = 1,2,\ldots,n-3)\\ (j = i+3,\ i+4,\ldots,n)\end{array} \tag{6.5.9}$$

To compute all the subdiagonal elements of $\underline{W}$:

$$w_{ij} = w_{ji} \prod_{k=j}^{(i-1)} (-b_k) \quad , \ j=1,2,\ldots,n-1,\ i=j+1,j+2,\ldots,n \tag{6.5.10}$$

To compute all the diagonal elements of $\underline{W}$:

$$w_{ii} = -\frac{1}{b_{i-1}}\left[w_{i(i-1)} - b_{i-1} - w_{i(i-2)}\right], i=3,4,\ldots,n \tag{6.5.11}$$

Let us also recall the common ratio property [For convenience, we call thus the property in (6.4.9) and (6.4.11) satisfied respectively by the superdiagonal elements and the subdiagonal elements of $\underline{W}$] discussed in section 6.4.

It is true that the equations (6.5.6) - (6.5.10) do not explicitly tell how the computation of w_{ij}'s is carried out from the knowledge of the b_i's. First let us assume that w_{11} has been computed (we shall explain

how this is done a little later). We shall now carefully explain how the other elements of $\underline{W}$ are computed.

Find w_{12}, w_{21} and w_{22} using (6.5.6) and w_{13} using (6.5.8) with $i=1$. Compute all the other elements in the first row using (6.5.9) i.e.

$$w_{1j} = -\frac{1}{b_{j-1}} w_{1(j-1)} - w_{1(j-2)}, \quad j = 4,5,\ldots,n \quad (6.5.12)$$

and then find all the elements in the first column using (6.5.10) i.e.

$$w_{j1} = w_{1j} (-1)^{j-1} (b_1 b_2 b_3 \ldots b_{j-1}), \quad j=3,4,\ldots,n \quad (6.5.13)$$

From now on there is a nice pattern of computation. Find w_{23} using (6.5.7). As $\frac{w_{23}}{w_{13}} = \frac{w_{24}}{w_{14}} \ldots = \frac{w_{2n}}{w_{1n}}$ (common ratio property), just knowing w_{23} and the first row elements, determine all the elements to the right of w_{23} in the second row. Use (6.5.10) to find w_{32}. i.e. $w_{32} = -b_2 w_{23}$. As $\frac{w_{32}}{w_{31}} = \frac{w_{42}}{w_{41}} = \ldots = \frac{w_{n2}}{w_{n1}}$, find all the elements in the second column below w_{32}. Now find w_{33} from (6.5.11). Continue this process of computation exactly in a similar pattern until the element w_{nn} is computed.

The above described computational scheme is recursive in nature and this is well-suited for use on computer. Let us next see how we can compute w_{11}. From

$$\underline{W} = \underline{I} + 2(\underline{B} - \underline{I})^{-1}$$

we obtain

$$w_{11} = 1 + \frac{2\left[\text{cofactor of } 11 \text{ element of } (\underline{B} - \underline{I})\right]}{\left[\text{Determinant of } (\underline{B} - \underline{I})\right]}$$

It is not very difficult to see (Refer Appendix I) that

$$w_{11} = 1 + 2B(n-1)/B(n) \tag{6.5.14}$$

Now $B(n)$ and $B(n-1)$ can be easily obtained from the recursive relations (Appendix I)

$$\left.\begin{aligned} B(o) &= 1 \\ B(1) &= -(1 + b_1) \\ B(k) &= -B(k-1) + b_{n-k+1}\, B(k-2) \\ & \qquad k = 2,3,\ldots,n \end{aligned}\right\} \tag{6.5.15}$$

So we infer that the rules for determining w_{ij}'s from b_i's discussed in section 6.3 are useful for hand calculation whereas the above described recursive computational scheme is well-suited for use on computer.

In the rest of this section, attempt is made to relate the Schwarz elements b_i's with the elements of the matrix $\underline{A}$ which is assumed to be in the phase-variable form and to relate the elements of $\underline{W}$, w_{ij}'s , with those of $\underline{A}$. It is also speculated that such relations may be expressed through the elements of the Jury table of the characteristic polynomial of $\underline{A}$. Even though complete success is not obtained in this investigation, it is hoped

that the results to be presented will be of considerable use for further investigation. Before proceeding further let us recapitulate that in the continuous case, the elements of the Schwarz matrix are related to the elements of the characteristic polynomial (or the elements of the corresponding phase-variable form) through the Hurwitz determinants or through the first column elements of the Routh array. Thus intuitively it is felt that the elements of $\underline{W}$ matrix may be related to those of $\underline{A}$ matrix through Schur-Cohn determinants or Jury table elements.

Let the characteristic equation of $\underline{A}$ be given by (6.5.1). Let J_{ij} denote the ij-th element of the Jury table of (6.5.1). The following results are obtained for $n = 2,3$ and 4.

<u>n = 2</u>

$$\left.\begin{aligned} b_2 &= \frac{2(1 - a_2)}{(1 + a_1 + a_2)} \\ b_1 &= \frac{(1 - a_1 + a_2)}{(1 + a_1 + a_2)} \end{aligned}\right\} \qquad (6.5.16)$$

<u>n = 3</u>

$$\left.\begin{aligned} b_3 &= \frac{(3 + a_1 - a_2 - 3a_3)}{(1 + a_1 + a_2 + a_3)} \\ b_2 &= \frac{(8 - 8a_2 - 8a_3^2 + 8a_1a_3)}{(1+a_1+a_2+a_3)(3+a_1-a_2-3a_3)} \\ b_1 &= \frac{(1 - a_1 + a_2 - a_3)}{(3 + a_1 - a_2 - 3a_3)} \end{aligned}\right\} \qquad (6.5.17)$$

<u>n = 4</u>

$$b_4 = \frac{(4 + 2a_1 - 2a_3 - 4a_4)}{(1 + a_1 + a_2 + a_3 + a_4)}$$

$$b_3 = \frac{\left[\begin{array}{l}(20 - 20a_4^2 + 10a_1 - 18a_3 - 12a_2 - 2a_3^2 + 2a_2a_3 \\ - 10a_3a_4 + 12a_2a_4 + 18a_1a_4 + 2a_1^2 - 2a_1a_2)\end{array}\right]}{(1 + a_1 + a_2 + a_3 + a_4)(4 + 2a_1 - 2a_3 - 4a_4)}$$

$$b_2 = \frac{\left[\begin{array}{l}(64 - 64a_4^2 - 64a_2 - 64a_3^2 + 128a_2a_4 + 64a_1a_3 \\ + 64a_1a_3a_4 - 64a_1^2a_4 - 64a_4 + 64a_4^3 - 64a_2a_4^2)\end{array}\right]}{(4 + 2a_1 - 2a_3 - 4a_4)\left[\begin{array}{l}(20 - 20a_4^2 + 10a_1 - 18a_3 - 12a_2 \\ -2a_3^2 + 2a_3a_2 - 10a_3a_4 + 12a_2a_4 \\ + 18a_1a_4 + 2a_1^2 - 2a_1a_2)\end{array}\right]}$$

$$b_1 = \frac{(1 - a_1 + a_2 - a_3 + a_4)\ (4 + 2a_1 - 2a_3 - 4a_4)}{\left[\begin{array}{l}(20 - 20a_4^2 + 10a_1 - 18a_3 - 12a_2 - 2a_3^2 + 2a_2a_3 \\ -10a_3a_4 + 12a_2a_4 + 18a_1a_4 + 2a_1^2 - 2a_1a_2)\end{array}\right]}$$

... (6.5.18)

For higher order systems the corresponding expressions become more and more complicated. In the above, b_i's are related to a_i's directly. We shall now express these relations through the Jury table elements. We shall denote the characteristic polynomial of $\underline{A}$ by F(z).

<u>n = 2</u>

$$\left.\begin{aligned} b_2 &= 2\,(J_{21} - J_{11})/F(1) \\ b_1 &= (-1)^2\, F(-1)/F(1) \end{aligned}\right\} \tag{6.5.19}$$

<u>n = 3</u>

$$\left.\begin{aligned} b_3 &= \frac{3(J_{21} - J_{11}) + (J_{22} - J_{12})}{F(1)} \\ b_2 &= \frac{4\left[2(J_{41} - J_{31})\right]}{F(1)\left[3(J_{21} - J_{11}) + (J_{22} - J_{12})\right]} \\ b_1 &= (-1)^3\, \frac{F(-1)}{F(1)}\; \frac{1}{b_3} \end{aligned}\right\} \tag{6.5.20}$$

For n=4, n=5 etc. , the expressions for b_n and b_1 show some pattern whereas for others, it is not possible at present to identify any such pattern. Expressions for b_n and b_1 for n=4 and n=5 are given below:

<u>n = 4</u>

$$\left.\begin{aligned} b_4 &= \frac{4(J_{21} - J_{11}) + 2(J_{22} - J_{12})}{F(1)} \\ b_1 &= (-1)^4\, \frac{F(-1)}{F(1)}\; \frac{1}{b_3} \end{aligned}\right\} \tag{6.5.21}$$

<u>n = 5</u>

$$\left.\begin{aligned} b_5 &= \frac{5(J_{21} - J_{11}) + 3(J_{22} - J_{12}) + (J_{23} - J_{13})}{F(1)} \\ b_1 &= (-1)^5\, \frac{F(-1)}{F(1)}\; \frac{1}{b_3 b_5} \end{aligned}\right\} \tag{6.5.22}$$

Let us now express the elements of $\underline{W}$ in terms of the a_i's at least in simple cases.

<u>n = 2</u>

$$\underline{W} = \begin{bmatrix} (a_2 - a_1 - 1)/2 & -(1 + a_1 + a_2)/2 \\ (1 - a_1 + a_2)/2 & (1 - a_1 - a_2)/2 \end{bmatrix} \tag{6.5.23}$$

<u>n = 3</u>

$$\left.\begin{aligned}
w_{11} &= \frac{a_3 - a_1}{2} - \frac{2(1 - a_2 - a_3^2 + a_1 a_3)}{(3 + a_1 - a_2 - 3a_3)} \\
w_{12} &= (a_3 - a_1)/2 - 1 \\
w_{13} &= -(1 + a_1 + a_2 + a_3)/4 \\
w_{21} &= 1 + \frac{(a_3 - a_1)}{2} - \frac{2(1 - a_2 - a_3^2 + a_1 a_3)}{(3 + a_1 - a_2 - 3a_3)} \\
w_{22} &= (a_3 - a_1)/2 \\
w_{23} &= -(1 + a_1 + a_2 + a_3)/4 \\
w_{31} &= \left\{1 + \frac{(a_3 - a_1)}{2} - \frac{2(1 - a_2 - a_3^2 + a_1 a_3)}{(3 + a_1 - a_2 - 3a_3)}\right\} \\
&\quad \left\{\frac{2(1 - a_2 - a_3^2 + a_1 a_3)}{(-1 + (a_3 - a_1)/2)\,(3 + a_1 - a_2 - 3a_3)}\right\} \\
w_{32} &= \frac{2(1 - a_2 - a_3^2 + a_1 a_3)}{(3 + a_1 - a_2 - 3a_3)} \\
w_{33} &= -a_3 + \frac{2(1 + a_1 a_3 - a_2 - a_3^2)}{(3 + a_1 - a_2 - 3a_3)}
\end{aligned}\right\} \tag{6.5.24}$$

It may be recalled (section 6.4) that only n elements of $\underline{W}$ are independent . Suppose we consider these n elements to be those in the first row of $\underline{W}$. Then the following results are obtained.

n = 2

$$\left.\begin{aligned} w_{12} &= -\frac{F(1)}{2} \\ w_{11} &= \frac{J_{11} - J_{22}}{2} - 1 \end{aligned}\right\} \tag{6.5.25}$$

n = 3

$$\left.\begin{aligned} w_{13} &= -\frac{F(1)}{2^2} \\ w_{12} &= \frac{J_{11} - J_{22}}{2} - 1 \\ w_{11} &= \frac{J_{11} - J_{22}}{2} - \frac{2(J_{41} - J_{31})}{3(J_{21} - J_{11}) + (J_{22} - J_{12})} \end{aligned}\right\} \tag{6.5.26}$$

Even though such results are not presently available for higher order systems, we observe that even in higher order systems

$$w_{1n} = -\frac{F(1)}{2^{(n-1)}} \tag{6.5.27}$$

In what follows, an attempt is made to express in general the first row elements of $\underline{W}$ in terms of the elements a_i's of $\underline{A}$. Unfortunately, such an expression involves some of the Routh array elements of the bilinear

transformed system. However, the expression obtained is quite interesting and may be of use for further investigation.

The matrix $\underline{A}$ is assumed to be in phase-variable form. The expression to be given is obtained after generalizing the results from systems of order upto seven. The method given in the beginning of this section is used for obtaining the results for low order systems. Before stating the results, let us define the following quantities.

$$\underline{W}^{(1)} = (w_{11}, w_{12}, \ldots, w_{1n})^T \quad , \underline{a} = (a_1, a_2, \quad , a_n)^T \qquad (6.5.28)$$

$$\left.\begin{array}{lll} \underline{F} & = & [F_{ij}]_{nxn} \\ \text{with} & & \\ F_{1j} & = & j \; , \quad j = 1,2,\ldots,n \\ F_{ij} & = & 0 \; , \quad i > j \\ F_{ij} & = & W_{i(j-1)} + 2F_{(i-1)(j-1)}, \; i \leq j \end{array}\right\} \qquad (6.5.29)$$

Note that $\underline{F}$ is an upper triangular matrix.

$$\underline{\hat{F}} = \left[\begin{array}{ccccc} 0 & 0 & \ldots & 0 & p_n/2 \\ \hline & & \underline{F} & & \end{array}\right]_{(n+1)xn} \qquad (6.5.30)$$

where

$$p_n = 1 + a_1 + a_2 + a_3 + \cdots + a_n$$

$$\left.\begin{aligned}
\underline{S} &= \left[S_{ij}\right]_{n\times n} \\
&\text{with} \\
S_{i(n-i+1)} &= -2\ ,\quad i = 1,2,\ldots,n \\
S_{ij} &= 0\ ,\quad j > (n-i+1),\ i > 1 \\
\left|S_{ij}\right| &= \left|S_{i(j+1)}\right| + \left|S_{(i-1)(j+1)}\right|
\end{aligned}\right\} \qquad (6.5.31)$$

Sign of the elements of $\underline{S}$ in a row alternate starting from the cross-diagonal entries and moving leftwards in that row. Note that $\underline{S}$ is a cross-diagonal matrix (i.e. elements lying below the cross-diagonal are zeros) with all the cross-diagonal elements equal to -2.

$$\hat{\underline{S}} = \left[\begin{array}{c|c} \underline{S} & \begin{matrix}1\\0\\ \cdot\\ \cdot\\ \cdot\\0\\0\end{matrix}\end{array}\right]_{n\times(n+1)} \qquad (6.5.32)$$

$$\left.\begin{aligned}
\underline{T}_k &= \left[T_{ij}\right]_{(n-2k)\times(n-2k)} \\
&\text{Let} \\
m &= n - 2k \\
&\text{Then} \\
T_{i(m-i+1)} &= 1\ ,\quad i = 1,2,\ldots,m \\
T_{ij} &= 0\ ,\quad j > m-i+1,\ i > 1 \\
\left|T_{1j}\right| &= \left(\left|T_{1(j+1)}\right| + m-j+1\right),\ j = m-1,\ m-2,\ldots,1 \\
\left|T_{ij}\right| &= \left|T_{(i-1)(j+1)}\right| + \left|T_{i(j+1)}\right|,\ j < m-i+1,\ i < (m-1)
\end{aligned}\right\}$$

$$\ldots\ (6.5.33)$$

sign of the elements of $\underline{T}_k$ in a row alternate starting from the cross-diagonal elements and moving leftwards in that row. Note that $\underline{T}_k$ also is a cross-diagonal matrix with the cross-diagonal elements equal to 1.

$$\hat{\underline{T}}_k = \begin{array}{c} \\ n-2k \\ 2k \end{array} \begin{array}{c} \begin{array}{cc} n-2k & 2k+1 \end{array} \\ \left[\begin{array}{c|c} \underline{T}_k & \underline{0} \\ \hline \underline{0} & \underline{0} \end{array}\right]_{n\times(n+1)} \end{array} \qquad (6.5.34)$$

$$\underline{a} = (a_{n-1}, a_{n-2}, \ldots, a_1, 1)^T \qquad (6.5.35)$$

$$\hat{\underline{R}}_k = \begin{array}{c} \\ n-2k \\ 2k \end{array} \begin{array}{c} \begin{array}{cc} n-2k & 2k \end{array} \\ \left[\begin{array}{c|c} \underline{R}_k & \underline{0} \\ \hline \underline{0} & \underline{0} \end{array}\right]_{n\times n} \end{array} \qquad (6.5.36)$$

where $\underline{R}_k$ is an $(n-2k) \times (n-2k)$ diagonal matrix, diagonal entries being determined by the Routh array elements of $\underline{C}_d$ (see section 6.2).

With the above quantities, we are now ready to present the relationship between $\underline{w}$ and $\underline{a}$.

$$\underline{w} = \frac{1}{2(n-1)} \left\{ \hat{\underline{S}} + 2 \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \hat{\underline{R}}_k \, \hat{\underline{T}}_k \right\} \hat{\underline{F}} \, \underline{a} \qquad (6.5.37)$$

where $\underline{R}_o = \underline{0}$ and $[k]$ means the largest integral contained in k. The equation (6.5.37) may be viewed as

representing the transformation between $\underline{w}$ and $\underline{a}$. This viewpoint becomes more apparent if we rewrite (6.5.37) as

$$\underline{w} = \underline{K}\ \underline{a}$$

$$\underline{K} = \frac{1}{2^{(n-1)}}\left\{\underline{\hat{S}} + 2\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \underline{\hat{R}}_k\ \underline{\hat{T}}_k\right\}\underline{\hat{F}} \qquad (6.5.38)$$

Note carefully that the transformation matrix $\underline{K}$ involves $\underline{\hat{S}}$, $\underline{\hat{T}}_k$ ($k = 0,1,\ldots,\left[\frac{n-1}{2}\right]$) and $\underline{\hat{F}}$, that these matrices are formed quite simply and that they contain just numbers i.e. they do not depend upon the system. The dependence on the system comes through $\underline{\hat{R}}_k$ in (6.5.38).

Let us illustrate these ideas by a simple example.

Example. For $n = 4$, determine $\underline{K}$.

$$\underline{K} = \frac{1}{8}\left(\underline{\hat{S}} + 2\underline{\hat{R}}_1\ \underline{\hat{T}}_1\right)\underline{\hat{F}}$$

$$p_4 = 1 + a_1 + a_2 + a_3 + a_4$$

The relation $\underline{w} = \underline{K}\ \underline{a}$ is illustrated in the next page.

Elements of $\underline{R}_k$ in (6.5.36) may also be expressed in general.

$$\underbrace{\begin{bmatrix} w_{11}\\ w_{12}\\ w_{13}\\ w_{14}\end{bmatrix}}_{\underline{w}} = \underbrace{\frac{1}{8}}_{2^3}\ \underbrace{\overbrace{\left[\begin{array}{cccc|c} 2 & -2 & 2 & -2 & 1\\ -6 & 4 & -2 & 0 & 0\\ 6 & -2 & 0 & 0 & 0\\ -2 & 0 & 0 & 0 & 0\end{array}\right]}^{\underline{S}}}_{\hat{\underline{S}}} + 2$$

$$\underbrace{\overbrace{\left[\begin{array}{cc|cc} R_{32}/R_{31} & 0 & 0 & 0\\ 0 & R_{22}/R_{21} & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]}^{\underline{R}_1}}_{\hat{\underline{R}}_1}\ \underbrace{\overbrace{\left[\begin{array}{cc|ccc} -3 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]}^{\underline{T}_1}}_{\hat{\underline{T}}_1}$$

$$\underbrace{\left[\begin{array}{cccc} 0 & 0 & 0 & p_4/2\\ \hline 1 & 2 & 3 & 4\\ 0 & 2 & 6 & 12\\ 0 & 0 & 4 & 16\\ 0 & 0 & 0 & 8\end{array}\right]}_{\underline{F},\ \hat{\underline{F}}}\ \underbrace{\begin{bmatrix} a_3\\ a_2\\ a_1\\ 1\end{bmatrix}}_{\underline{a}}$$

$\underline{w} = \underline{K}\ \underline{a}$ Illustrated For $n = 4$

6.6 PROBLEMS FOR FUTURE INVESTIGATION

Even while discussing in the previous sections, several problems for further investigation have been posed. In this section, in addition to presenting these problems together in one place, the problem of establishing a link between Jury's stability criteria [48, 49] and Liapunov's second method* will be considered.

First, a general proof may be given for ensuring that the negative semidefiniteness of $\underline{Q}$ in (6.3.10) implies the asymptotic stability of the system in (6.2.9). Second, the relations between the elements of the $\underline{W}$ matrix and those of the $\underline{A}$ matrix may be established in general either through the Schur-Cohn determinants or through the elements of the Jury table. Third, some more uses for this canonical matrix $\underline{W}$ may be found. For example, analogous to the Schwarz matrix being used for performance measure evaluation in the continuous case, $\underline{W}$ also may be put to such a use.

Finally, let us consider the problem of establishing a link between the Jury's stability constraints and Liapunov's second method. In section 6.3, it has been found [See equations (6.3.25) - (6.3.27)] that the application of the second method of Liapunov requires, for the asymptotic stability of the system, the following conditions to be satisfied by the elements of $\underline{W}$:

*Jury [48] while replying to the discussion of Parks has posed the same problem.

$$(-1)^{k+1} \quad \frac{w_{1k}}{w_{k1}} \quad 0 \ , \ k = 1,2,\ldots,n-1 \tag{6.6.1}$$

$$\frac{1 - w_{n(n-1)} + w_{nn}}{1 - w_{nn}} \quad 0 \tag{6.6.2}$$

Let us see how these constraints can be shown to be equivalent to the Jury's stability constraints [49], for the cases when $n=2$ and $n=3$.

<u>$n = 2$</u>

From (6.5.23) it is not very difficult to see that

$$\left.\begin{aligned} a_1 &= -(w_{11} + w_{22}) \\ \text{and} \quad a_2 &= (w_{11} w_{22} - w_{12} w_{21}) \end{aligned}\right\} \tag{6.6.3}$$

Also from

$$\underline{B}\,\underline{W} = \underline{B} + \underline{W} + \underline{I}$$

we get (see section 6.4)

$$\left.\begin{aligned} w_{21} &= 1 + w_{11} \\ w_{22} &= 1 + w_{12} \end{aligned}\right\} \tag{6.6.4}$$

Jury's stability constraints for this problem are:

$$F(1) > 0 \ , \quad F(-1) > 0 \quad \text{and} \quad \delta_1 < 0 \tag{6.6.5}$$

Now

$$F(1) > 0 \rightarrow 1 + a_1 + a_2 > 0 \tag{6.6.6}$$

From (6.6.3), (6.6.4) and (6.6.6), after some manipulations

we obtain

$$1 - w_{22} > 0 \tag{6.6.7}$$

Similarly $F(-1) > 0$ may be shown to be equivalent to

$$1 + w_{11} > 0 \tag{6.6.8}$$

From (6.6.7) and (6.6.8) we get

$$\frac{1 - w_{22}}{1 + w_{11}} > 0$$

In view of (6.6.4) this means

$$-w_{12}/w_{21} > 0 \tag{6.6.9}$$

Now

$$\delta_1 < 0 \Rightarrow a_2^2 - 1 < 0 \Rightarrow (a_2 + 1)(a_2 - 1) < 0 \tag{6.6.10}$$

Owing to (6.6.3) and (6.6.4)

$$a_2 + 1 = w_{21} - w_{12} \tag{6.6.11}$$

Combining (6.6.9) and (6.6.11)

$$a_2 + 1 > 0$$

This, in view of (6.6.10), implies

$$a_2 - 1 < 0 \tag{6.6.12}$$

Substituting (6.6.2) and (6.6.4) in (6.6.12) results in

$$1 - w_{21} + w_{22} > 0 \tag{6.6.13}$$

From (6.6.7) and (6.6.13) we obtain

$$\frac{1 - w_{21} + w_{22}}{1 - w_{22}} > 0 \qquad (6.6.14)$$

Note that (6.6.9) and (6.6.14) are exactly same as those obtained by putting $n=2$ in (6.6.1) and (6.6.2).

<u>$n = 3$</u>

In the above example, we started with Jury's stability constraints and then after many manipulations, we showed that these are equivalent to the stability constraints obtained by Liapunov's second method. Here, let us proceed in the reverse way. For $n=3$, our stability constraints are:

$$w_{12}/w_{21} < 0 \qquad (6.6.15)$$

$$w_{13}/w_{31} > 0 \qquad (6.6.16)$$

$$\frac{1 - w_{32} + w_{33}}{1 - w_{33}} > 0 \qquad (6.6.17)$$

Using the results of (6.5.24) it can be shown that (6.6.15) - (6.6.17) imply

$$\frac{3 + a_1 - a_2 - 3a_3}{1 - a_1 + a_2 - a_3} > 0 \qquad (6.6.18)$$

$$\left.\begin{array}{l} F(1) > 0 \quad , \quad F(-1) < 0 \\ a_3^2 - 1 < a_3 a_1 - a_2 \\ 1 - a_3 > 0 \end{array}\right\} \qquad (6.6.27)$$

Jury's constraints [48] are exactly similar to those in (6.6.27) except for the fact that he gets in addition to these

$$1 + a_3 > 0$$

(Actually he gives a condition $|a_3| < 1$. This would imply $1 - a_3 > 0$, $1 + a_3 > 0$). Does this mean that the condition $1 + a_3 > 0$ is not required ? Of course, this needs further investigation. We end this section by saying that it may be possible in general to establish a link between Jury's stability constraints and the second method of Liapunov.

CHAPTER - VII

CONCLUSIONS

The canonical form introduced by Schwarz serves as a useful way of characterizing the performance of linear time-invariant continuous-time systems. It has several useful properties which have been listed in this report. It is also shown to be useful in simplifying large dynamic systems and in deriving a canonical form for linear time-invariant discrete-time systems. The problem of transforming any given system to this canonical form has been dealt with very systematically. A Vandermonde-like matrix which transforms the Schwarz form to the Jordan form has also been derived. The Schwarz canonical form has been extended to multivariable systems also. A simple network interpretation has been given to the Schwarz canonical form.

The method for simplifying large dynamic systems investigated in this thesis is a matrix generalization of the associated transfer function introduced by Gustafson [25] . However, the approximation made is very much influenced by the type of inputs applied to the system. It may be possible to devise a method for simplifying large systems, which is optimal with respect to some particular input. This may call for the application

this direction has been recently made by Meier and Luenberger [38].

A new canonical form for linear time-invariant discrete-time systems introduced in this investigation has many useful properties similar to the Schwarz form in the continuous case. However, the investigation remains somewhat inconclusive since the relations between the canonical matrix elements and the Jury table elements are not generalized. If this is done, the Jury table test for stability may be proved through the second method of Liapunov.

Throughout the presentation of the results in this thesis, wherever necessary, problems which require further investigation have been posed. A brief summary of these problems will be given here.

1. A technique of modeling the physical system may be devised to yield the system description directly in the Schwarz form.
2. The Schwarz canonical form for multivariable systems may be investigated further and its practical uses may be determined.
3. For the simplification procedure introduced in this thesis, estimates of error between the actual and the approximate system responses may be determined. Some

difficulties associated with systems with complex eigenvalues may be resolved. It may be possible to find out a performance measure which becomes optimum for this simplification procedure. The problem of extending this method to multivariable systems also needs further investigation.

4. In connection with the proposed canonical form for linear time-invariant discrete-time systems, the problem of relating its elements to Jury table needs further investigation.

5. In the case of interconnected systems, the investigation of the stability of the entire system through that of subsystems has been carried out recently [50]. It is interesting to ask whether there will be any simplification if the individual subsystems are in Schwarz form. It may be possible to construct a Liapunov function for the entire system through the Liapunov functions of individual subsystems at least for simple modes of interconnection pattern.

APPENDIX - I

INVERSE OF TRIDIAGONAL MATRICES

Let $\underline{A}$ be a tridiagonal matrix. That is, if

$$\underline{A} = \left[a_{ij}\right]_{n\times n}$$

then

$$a_{ij} = 0 \quad \text{when} \quad j \neq i \quad \text{or} \quad j \neq i-1 \quad \text{or} \quad j \neq i+1$$

Here, it is aimed to state the general expressions for the elements of $\underline{A}^{-1}$ in terms of a_{ij}'s explicitly. These expressions are obtained by generalizing the results got from low order examples. Let

$$\underline{C} = \left[c_{ij}\right]_{n\times n} = \underline{A}^{-1}$$

The results will be stated now for three cases.

1. When $\underline{A}$ is any tridiagonal matrix
2. When $\underline{A} = \underline{B}$ where $\underline{B}$ is the Schwarz matrix
3. When $\underline{A} = (\underline{B} - \underline{I})$

<u>Case 1</u>: The elements C_{ij}'s are given by:

$$C_{ii} = \frac{1}{B(n)} \; F(i-1)\, B(n-i), \; i = 1,2,\ldots,n$$

$$C_{ij} = \begin{cases} \dfrac{1}{B(n)} \; F(j-1)\, B(n-i)\, (-1)^{i+j} \left[\prod_{k=j}^{i-1} a_{(k+1)k}\right], & i > j \\[2ex] \dfrac{1}{B(n)} \; F(i-1)\, B(n-j)\, (-1)^{i+j} \left[\prod_{k=i}^{j-1} a_{k(k+1)}\right], & j > i \end{cases}$$

$$i \neq j, \quad i,j = 1,2,\ldots,n$$

where

$$B(0) = 1$$

$$B(1) = a_{nn}$$

$$B(k) = a_{(n-k+1)(n-k+1)} B_{(k-1)} - a_{(n-k+2)(n-k+1)} \times a_{(n-k+2)(n-k+2)} B(k-2)$$

$$k = 2,3,\ldots,n$$

and

$$F(0) = 1$$

$$F(1) = a_{11}$$

$$F(k) = a_{kk} F(k-1) - a_{k(k-1)} a_{(k-1)k} F(k-2)$$

$$k = 2,3,\ldots,n \qquad \ldots (A.1.1)$$

Note that $B(k)$ is the k-th ascending principal minor and $F(k)$ is the k-th descending principal minor. Clearly

$$F(n) = B(n)$$

Case 2: When $\underline{A} = \underline{B}$

$$C_{ii} = \begin{cases} 0 & i \text{ even} \\ \frac{1}{B(n)} F(i-1)\, B(n-i), & i \text{ odd} \end{cases}$$

$$i = 1,2,\ldots,n$$

$$C_{ij}\underset{i\neq j}{=}\begin{cases} 0 & j \text{ even} \\ \dfrac{1}{B(n)}\left[\prod_{k=j}^{(i-1)}(-b_k)\right] F(j-1)\, B(n-i)\,(-1)^{i+j} & j \text{ odd} \end{cases} \quad i>j$$

$$\begin{cases} 0 & i \text{ even} \\ \dfrac{1}{B(n)} F(i-1)\, B(n-j)\,(-1)^{i+j} & i \text{ odd} \end{cases} \quad j>i$$

$$i,j=1,2,\ldots,n$$

where

$$F(k) = b_{k-1}\, F(k-2),\ k = 2,3,\ldots,n-1$$
$$F(0) = 1$$
$$F(1) = 0$$

and

$$B(k) = b_{n-k+1}\, B(k-2),\ k = 2,3,\ldots,n$$
$$B(0) = 1$$
$$B(1) = -b_n \qquad \ldots\ (A.1.2)$$

Note here also that $F(n) = B(n)$

Case 3:

$$\underline{A} = (\underline{B} - \underline{I})$$

$$C_{ii} = \frac{1}{B(n)} F(i-1)\, B(n-i)\ ,\quad i = 1,2,\ldots,n$$

$$C_{ij} = \begin{cases} \dfrac{1}{B(n)} F(j-1)\, B(n-i)\,(-1)^{i+j}\left[\prod_{k=j}^{i-1}(-b_k)\right], & i>j \\ \dfrac{1}{B(n)} F(i-1)\, B(n-j)\,(-1)^{i+j}, & j>i \end{cases}$$

$$i\neq j,\quad i,j = 1,2,\ldots,n$$

where

$$F(0) = 1$$

$$F(1) = -1$$

$$F(k) = -F(k-1) + b_{k-1}\, F(k-2)$$

$$k = 2,3,\ldots,(n-1),n$$

and

$$B(0) = 1$$

$$B(1) = -(1 + b_n)$$

$$B(k) = -B(k-1) + b_{n-k+1}\, B(k-2)$$

$$k = 2,3,\ldots,(n-1),n \qquad (A.1.3)$$

Note that $B(n) = F(n)$.

APPENDIX - II

NORMALIZATION PROCEDURES

Two normalization procedures are discussed here.

I. Consider the transfer function

$$G'(s) = \frac{x(s)}{v(s)} = \frac{g}{C_0 s^n + C_1 s^{n-1} + \cdots + C_{n-1} s + C_n} \qquad (A.2.1)$$

Divide the denominator and numerator of the expression in the right hand side of (A.2.1) by C_0. Define

$$\left.\begin{aligned} g/C_0 &= b \\ \frac{C_i}{C_0} &= a_i\,, \quad i = 1,2,\ldots,n \end{aligned}\right\} \qquad (A.2.2)$$

We get

$$G^{*}(s) = \frac{x(s)}{v(s)} = \frac{b}{s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n} \qquad (A.2.3)$$

Define

$$bv(s) = u(s) \qquad (A.2.4)$$

Then

$$G(s) \triangleq \frac{x(s)}{u(s)} \qquad \frac{1..}{s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n} \qquad (A.2.5)$$

$G(s)$ is the normalized transfer function used in section 5.2.

II. Consider the differential equation

$$C_0 \frac{d^n x}{dt^n} + C_1 \frac{d^{n-1}x}{dt^{n-1}} + \dots + C_{n-1}\frac{dx}{dt} + C_n x = g\,u(t) \qquad (A.2.6)$$

Substituting $\tau = \omega t$ where $\omega = n\sqrt{\frac{C_n}{C_0}}$ in (A.2.6) and dividing the resulting equation both sides by C_n and replacing $\frac{C_i}{C_n}\omega^{n-i}$ by a_i $(i = 1,2,\dots,n-1)$ and $\frac{gu(t)}{C_n}$ by $v(t)$ yield

$$\frac{d^n x}{d\tau^n} + a_1 \frac{d^{n-1}x}{d\tau^{n-1}} + \dots + a_{n-1}\frac{dx}{d\tau} + x = v(t) \qquad (A.2.7)$$

Taking Laplace transforms on both sides of (A.2.7)

$$F(s) = \frac{x(s)}{v(s)} = \frac{1}{s^n + a_1 s^{n-1} + \dots + a_{n-1}s + 1} \qquad (A.2.8)$$

This is the normalized transfer function corresponding (A.2.6).

APPENDIX - III

NUMERATOR DYNAMIC SYSTEMS

It is desired to extend the method of simplifying large systems, proposed in chapter V, to systems with numerator dynamics. Consider the system given by

$$x^{(n)} + a_1 x^{(n-1)} + a_2 x^{(n-2)} + \dots + a_{n-1} x^{(1)} + a_n x$$

$$= d_o\ u^{(m)} + d_1\ u^{(m-1)} + \dots + d_m u \qquad \text{(A.3.1)}$$

where m can be at most equal to ℓ, where ℓ is the size of the simplified system. (If we were to work with the system (A.3.1) itself without worrying about any simplification, then m can be less than or equal to n).

Two methods will be given to reduce (A.3.1) to the required form (5.2.3). These methods are discussed in detail in [43].

<u>Method 1</u>:

The system given in (A.3.1) can be equivalently represented in a matrix form as under:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{bmatrix} u \quad \ldots \quad (A.3.2)$$

where

$$y_1 = x - C_o u \qquad (A.3.3)$$

and if $m = n$, then

$$\begin{bmatrix} d_o \\ d_1 \\ d_2 \\ d_3 \\ \cdot \\ \cdot \\ d_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} C_o \\ C_1 \\ C_2 \\ C_3 \\ \cdot \\ \cdot \\ C_n \end{bmatrix} \qquad (A.3.4)$$

If $m = \ell < n$, then $d_o = d_1 = d_2 = d_{(n-\ell-1)} = 0$. Thus knowing d_i's $(i = 0,1,\ldots,n)$ and a_i's $(i = 1,2,\ldots,n)$ we can find C_i's by inverting the $(n+1)x(n+1)$ matrix in (A.3.4) and premultiplying the vector in the left hand side of (A.3.4). Again another coordinate transformation can be applied to take the system description from the form in (A.3.2) to that in (5.2.3). Note also that the

initial conditions for the vector $\underline{y} = (y_1, y_2, \ldots, y_n)^T$ in (A.3.2) are given by:

$$\begin{bmatrix} y_1(o) \\ y_2(o) \\ y_3(o) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n(o) \end{bmatrix} = \begin{bmatrix} x(o) \\ \dot{x}(o) \\ \ddot{x}(o) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^{(n-1)}(o) \end{bmatrix} - \begin{bmatrix} C_0 & 0 & 0 & \ldots & 0 \\ C_1 & C_0 & 0 & \ldots & 0 \\ C_2 & C_1 & C_0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ C_{n-1} & C_{n-2} & C_{n-3} & \ldots & C_0 \end{bmatrix} \begin{bmatrix} u(o) \\ \dot{u}(o) \\ \ddot{u}(o) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u^{(n-1)}(o) \end{bmatrix} \qquad (A.3.5)$$

Method II:

By this method, the system in (A.3.1) can be equivalently represented in the form given in (5.2.3) where now the relation between x and $x_1, x_2, \ldots, x_n$ is given by:

$$x = b_o u + (d_n - a_n d_o)\, x_1 + (d_{n-1} - a_{n-1} d_o)\, x_2 + \cdots$$

$$\cdots + (d_2 - a_2 d_o)\, x_{n-1} + (d_1 - a_1 d_o) x_n \qquad (A.3.6)^*$$

To determine the initial conditions for (5.2.3), differentiate (A.3.6) successively (n-1) times each time substituting for $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ from (5.2.3). At $t = t_o$

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*Note that in writing down (A.3.6), it is assumed that 'm' appearing in (A.3.1) equals n. If $m = n$, then $d_o = d_1 = d_2 = \ldots = d_{n-1} = 0$.

(initial time), (A.3.6) and the (n-1) differentiated equations will give the necessary n equations to evaluate $x_1(t_o), x_2(t_o), \ldots, x_n(t_o)$.

As far as the simplification method suggested in this work is concerned, method II seems to be easier than method I. In section 5.5 a simple example system with numerator dynamics with $n = 4$ is reduced to a system of order two.

APPENDIX - IV

SIMPLIFICATION OF A SECOND ORDER SYSTEM

In section 2 of chapter V, it is stated that unless $b_{i+1} < b_i$, the comparison process must be performed till b_2 is compared with b_1. Suppose the comparison process has ended with b_2 being much larger than b_1. We infer that the given system may be reduced to a second order system. Can we set up a criterion to check whether this reduced second order system can be simplified any further (i.e. can it be reduced to a first order system) ? Obviously no one likes to reduce a very high order system to a first order system! Yet for completeness sake we will answer the above question.

It is very obvious that the second order system to be reduced to a first order system cannot have repeated or complex eigenvalues. Let us now state the required

criterion. Whenever $b_1 \gg b_2$ or $b_2 \gg b_1$ or $b_1 \doteq b_2$ but each of them being much larger than 1, infer that the given second order system can be reduced to a first order system.

Examples.

1) Given a system as in (5.2.3) with $a_1 = 101$ and $a_2 = 100$. Therefore, $b_2 = 101$ and $b_1 = 100$. Here, $b_1 = b_2$ and b_1 and $b_2 \gg 1$. So this system can be reduced to a first order system. In fact the eigenvalues of the given system are -1 and -100.

2) $b_2 = a_1 = 20.1$; $b_1 = a_2 = 2$. Here $b_2 \gg b_1$, In fact the eigenvalues of the system are -0.1 and -20.

3) $b_2 = a_1 = 1010$; $b_1 = 10{,}000$. Here $b_1 \gg b_2$. The eigenvalues of the system are -10 and -1000.

4) $b_2 = a_1 = 3$; $b_1 = a_2 = 2$. Here $b_1 \cong b_2$ and b_1 and b_2 are close to 1. Therefore no approximation can be made. (The eigenvalues of the original system are -1 and -2).

5) If the system has repeated or complex eigenvalues, this criterion fails.

   (i) $b_2 = a_1 = 200$; $b_1 = a_2 = 10{,}000$. Here $b_1 \gg b_2$. But yet eigenvalues of the system are -100 and -100.

(ii) $b_2 = a_1 = 1$; $b_1 = a_2 = 25.25$. Here $b_1 \gg b_2$. However, the eigenvalues of this system are $-0.5 \pm j\,5$.

So in conclusion it may be stated that whenever we have $b_2 \hat{=} b_1$ and both b_2 and b_1 are much larger than one, reduction can be performed. The reduced first order system has $b_1^* = \frac{b_1}{b_2}$. In other cases, it is necessary to find the eigenvalues $\left[\frac{-b_2 \pm \sqrt{b_2^2 - 4b_1}}{2} \right]$ and if they are sufficiently apart, perform the reduction.

APPENDIX - V

ELIMINATION OF REDUNDANT STATE VARIABLES THROUGH THE USE OF SCHWARZ CANONICAL FORM

When several subsystems are assembled to form the main system, it is quite possible for such an assembled system to have some redundant state variables [36, 44]. Presence of such redundant state variables can be easily detected when the main system equations are transformed to the Schwarz form. If the system is stable, then zero values of b_i's would indicate the presence of

redundant state variables. If there is one redundancy then b_1 will be zero. If two then both b_1 and b_2 will be zero, and so on. This statement can be very easily proved. Redundant state variables introduce zero eigenvalues. Presence of zero eigenvalues can be detected from zero values of bottom elements of the Routh array. That is, presence of one zero eigenvalue, for instance, results in $R_{n+1,1} = 0$. Expressing b_i's in terms of the Routh array elements

$$b_1 = \frac{R_{(n+1)1}}{R_{(n-1)1}}$$

$$b_2 = \frac{R_{n,1}}{R_{(n-2)1}}$$

Hence the proof.

The redundant state variables can be easily eliminated by discarding the first rows and columns of the Schwarz matrix and also of the transformation matrix $\underline{P}$. We shall illustrate this by means of a simple example. Consider a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

i.e. $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{f}\,u$

$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_1 & 0 & 1 & 0 \\ 0 & b_1+b_2 & 0 & 0 \end{bmatrix}$$

Let the transformed system be given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_1 & 0 & 1 & 0 \\ 0 & -b_2 & 0 & 1 \\ 0 & 0 & -b_3 & -b_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Suppose $b_1 = 0$. Then discard the first column and the first row of $\underline{B}$ and the corresponding variable z_1 has to be eliminated. This is illustrated below.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_1 & 0 & 1 & 0 \\ 0 & -b_2 & 0 & 1 \\ 0 & 0 & -b_3 & -b_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

The transformation matrix $\underline{P}$ is also to be changed similarly i.e. its first row and first column have to be discarded.

$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_1 & 0 & 1 & 0 \\ 0 & \underset{=b_2}{b_1+b_2} & 0 & 1 \end{bmatrix}$$

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